Singapore Open Mathematical Olympiad 2009

Junior Section

Tuesday, 2 June 2009 0930-1200 hrs

Important:
Answer ALL 35 questions.
Enter your answer sheet provided.
For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer
For the other short questions, write your answer in the answer sheet
No steps are needed to justify your answer
Each question carries 1 mark
No calculators are allowed.

Multiple Choice Questions

Question 1. Let C_1 and C_2 be distinct circles of radius 7 cm that are in the same plane and tangenr to each other. Find the number of circles of radius 26 cm in this plane that are tangent to both C_1 and C_2.

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** non of the above

Question 2. In the diagram below, the radius of quadrant ODA is 4 and the radius of quadrant OBC is 8. Given that $\angle COD = 30^0$, find the area of the shades region $ABCD$.

(A) 12π **(B)** 13π **(C)** 15π **(D)** 16π **(E)** non of the above

Question 3. Let k be a real number. Find the maximum value of k such that the following inequality holds:

$$\sqrt{x-2}+\sqrt{7-x}\geqslant k.$$

(A) $\sqrt{5}$ **(B)** 3 **(C)** $\sqrt{2}+\sqrt{3}$ **(D)** $\sqrt{10}$ **(E)** $2\sqrt{3}$

Question 4. Three circles of radius 20 are arranged with their respective centres A, B and C in a row. If the line WZ is tangent to the third circle, find the langth of XY.

(A) 30 **(B)** 32 **(C)** 34 **(D)** 36 **(E)** 38

Question 5. Given that x and y are both negative integers satisfying the equation $y = \frac{10x}{10-x}$, find the maximum value of y.

(A) -10 **(B)** -9 **(C)** -6 **(D)** -5 **(E)** non of the above

Question 6. The sequence a_n satisfy $a_n = a_{n-1} + n^2$ and $a_0 = 2009$. Find a_{50}.

(A) 42434 **(B)** 42925 **(C)** 44934 **(D)** 45029 **(E)** 45359

Question 7. Coins of the same size are arranged on a very large table (the infinite plane) such that each coin touches six other coins. Find the percetage of the plane that is covered by the coins.

(A) $\frac{20}{\sqrt{3}}\pi$ % **(B)** $\frac{50}{\sqrt{3}}\pi$ % **(C)** $16\sqrt{3}\pi$ % **(D)** $17\sqrt{3}\pi$ % **(E)** $18\sqrt{3}\pi$ %

Question 8. Given that x and y are real numbers satisfying the followwing equations:

$$x + xy + y = 2 + 3\sqrt{2} \quad \text{and} \quad x^2 + y^2 = 6,$$

find the value of $|x + y + 1|$.

(A) $1 + \sqrt{3}$ **(B)** $2 - \sqrt{3}$ **(C)** $2 + \sqrt{3}$ **(D)** $3 - \sqrt{2}$ **(E)** $3 + \sqrt{2}$

Question 9. Given that $y = (x - 16)(x - 14)(x + 14)(x + 16)$, find the minimum value of y.

(A) -896 **(B)** -897 **(C)** -898 **(D)** -899 **(E)** -900

Question 10. The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$ with the condition that $a < b < c < d$ is

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Short Questions

Question 11. There are two models of LCD television on sale. One is a Ò20 inchÒ stabdard modl while the other is a Ò20 inchÒ widescreen model. The ratio of length to the height of the standard mode is $4:3$, while that of the widescreen model is $16:9$. Television screens are measured by the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is $A:300$, find the value of A.

Question 12. The diagram below shows a pentagon (made up of region A and region B) and a rectangle (made up of region B and region C) that overlap. The overlapped region B is $\frac{3}{16}$ of the pentagon and $\frac{2}{9}$ of the rectangle. If the ratio of region A of the pentagon to region C of the rectangle is $\frac{m}{n}$ in its lowest term, find the value of $m+n$.

Question 13. 2009 students are taking a test which comprises ten true or fales questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

Question 14. The number of ways to arrange 5 boys and 6 giels in a row such that girls can be adjacent to other girls but boys cannot be adjacent to other boys is $6! \times k$. Find the value of k.

Question 15. ABC is a right-angled triangle with $\angle BAC = 90^0$. A square is constructed on the side AB and BC as shown. The area of the square $ABDE$ is 8 cm^2 and the area of the square $BCFG$ is 26 cm^2. Find the area of triangle DBG in cm^2.

Question 16. The sum of $\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\frac{1}{4\times 5\times 6}+\cdots+\frac{1}{13\times 14\times 15}+\frac{1}{14\times 15\times 16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m+n$.

Question 17. Given that $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$ and $a-b+2\neq 0$, find the value of $ab-a+b$.

Question 18. If $|x|+x+5y=2$ and $|y|-y+x=7$, find the value of $x+y+2009$.

Question 19. Let p and q represent two consecutive prime number. For some fixed integer n, the set $\{n-1, 3n-19, 38-5n, 7n-45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of n.

Question 20. Find the number of ordered pairs of positive intergers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0.$$

Question 21. Find the intergers part of

$$\frac{1}{\frac{1}{2003} + \frac{1}{2004} + \frac{1}{2005} + \frac{1}{2006} + \frac{1}{2007} + \frac{1}{2008} + \frac{1}{2009}}.$$

Question 22. The diagram below shows a rectangle $ABLJ$, where the area of ACD, $BCEF$, $DEIJ$ and FGH are 22 cm^2, 500 cm^2 and 22 cm^2 respectively. Find the area of HIK in cm^2.

Question 23. 23. Evaluate $\sqrt[3]{77 - 20\sqrt{13}} + \sqrt[3]{77 + 20\sqrt{13}}$.

Question 24. Find the number of integers in the set $\{1, 2, 3, \ldots, 2009\}$ whose sum of the digits is 11.

Question 25. Given that

$$x + (1 + x)^2 + (1 + x)^3 + \cdots + (1 + x)^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^2,$$

where each a_r is an integer, $r = 0, 1, 2, \ldots, n$.

Find the value of n such that

$$a_0 + a_2 + a_3 + a_4 + \cdots + a_{n-2} + a_{n-1} = 60 - \frac{n(n+1)}{2}.$$

Question 26. In the diagram, OAB is a triangle with $\angle AOB = 90^0$ and $OB = 13$ cm. $P\&Q$ are 2 points on AB such that $26AP = 22PQ = 11QB$. If the vertical height of $PQ = 4$ cm, find the area of the triangle OPQ in cm^2.

Question 27. Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 + kx^2 + 90x - 2009 = 0.$$

If $x_1x_2 = 49$, Find the value of k.

Question 28. Three sides OAB, OAC and OBC of tetrahedron $OABC$ are right-angled triangles, i.e. $\angle AOB = \angle AOC = \angle BOC = 90^0$. Given that $OA = 7$, $OB = 2$ and $OC = 6$, find the value of

$$(\text{ Area of } \Delta OAB)^2 + (\text{ Area of } \Delta OAC)^2 + (\text{ Area of } \Delta OBC)^2 + (\text{ Area of } \Delta ABC)^2.$$

Question 29. Find the least positive integer n for which $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction.

Question 30. Find the value of the smallest positive integer m such that the equation

$$x^2 + 2(m+5)x + (100m+9) = 0$$

has only integer solutions.

Question 31. In a triangle ABC, the length of the altitudes AD and BE are 4 and 12 respectively. Find the largest possible integer calue for the length of third altitude CF.

Question 32. A four-digit number consists of two distinct pairs of repeated digit (for example 2211, 2626 and 7007). Find the total number of such possible number that are divisible by 7 or 101 but not both.

Question 33. m and n are two positive integer satisfying $1 \leqslant m \leqslant n \leqslant 40$. Find the number of pairs of (m, n) such that their produc mn is divisible by 33.

Question 34. Using the digits 0, 1, 2, 3, and 4, find the number of 13-digit sequences that can be written so that the diffenrence between any two consecutive digits is 1.

Question 35. m and n are two positive integers of reverse order (for example 123 and 321) such that $mn = 1446921630$. Find the value of $m + n$.
