Hanoi Mathematical Society

Hanoi Open Mathematics Competition 2013

Junior Section

Sunday, March 24, 2013 14h00-17h00

Multiple Choice Questions

Question 1. Write 2013 as a sum of m prime numbers. The smallest value of m is:

(A): 2; (B): 3; (C): 4; (D): 1; (E): None of the above.

Answer: (A).

Since $2013 = 3 \times 671$ then 2013 is not a prime number. Hence $m \geq 2$. On the other hand, $2013 = 2 + 2011$ as a sum of 2 prime numbers. Thus, the smallest value of m is 2.

Question 2. How many natural numbers n are there so that $n^2 + 2014$ is a perfect square.

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Answer: (E).

Suppose that $n^2 + 2014$ is a perfect square, i.e. $n^2 + 2014 = m^2$, where $m \in \mathbb{N}^*$. It follows $(m-n)(m+n) = 2014$ and then at least one of $m-n$ and $m+n$ is even. Since $(m-n)+(m+n) = 2m$ is even then both $m+n$ and $m-n$ are even. Hence $(m-n) \times (m+n)$ is devisible by 4. It is imposible for 2014 is not divisible by 4. Thus, there are no natural numbers n so that $n^2 + 2014$ is a perfect square.

Question 3. The largest integer not exceeding $[(n+1)\alpha] - [n\alpha]$, where n is a natural number, $\alpha = \dfrac{\sqrt{2013}}{\sqrt{2014}}$, is:

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Answer: (E).

Let $a_n = [(n+1)\alpha] - [n\alpha]$, for $n = 0, 1, 2, \ldots$ From the inequalities $0 \leq a_n \leq [n\alpha + 1] - [n\alpha] = 1$ for every natural number n and a_n is an integer, it follows $a_n = [(n+1)\alpha] - [n\alpha] \in \{0, 1\}$ for every $n \in \mathbb{N}$. We prove that 0 is the largest integer not exceeding every $[(n+1)\alpha] - [n\alpha]$. Indeed, for $n=0$, we find $a_0=[\alpha]=0$. Hence, the largest integer not exceeding $[(n+1)\alpha]-[n\alpha]$, where n is a natural number and $\alpha=\dfrac{\sqrt{2013}}{\sqrt{2014}}$ must be 0.

Question 4. Let A be an even number but not divisible by 10. The last two digits of A^{20} are:

(A): 46; (B): 56; (C): 66; (D): 76; (E): None of the above.

Answer: (D).

Since A is even then $A=2n$, $n\in\mathbb{N}$. It follows $A^{20}=(2n)^{20}=\left(4n^2\right)^{10}\Rightarrow A^{20}\,\vdots\,4$.

On the other hand, A is not divisible by 10,

$$\left[\begin{array}{lcl} A&=&5k\pm 1\\ A&=&5k\pm 2\end{array}\right.$$

If $A=5k\pm 1$ then $A^{20}=(5k\pm 1)^{20}=(5k)^{20}+20.(5k)^{19}+20.(5k)^{18}+....+20.5k+1$, hence $A^{20}\equiv 1 \pmod{25}$.

If $A=5k\pm 2$, then $A^{20}=25q+2^{20}=25q+(1025-1)^2$, hence $A^{20}\equiv 1 \pmod{25}$.

Thus, $A^{20}\equiv 1 \pmod{25}$ for every A and the last two digits of A^{20} are in $\{01;26;51;76\}$. Since A^{20} is divisible by 4 then the last two digits of A^{20} are 76.

Question 5. The number of integer solutions x of the equation below

$$(12x-1)(6x-1)(4x-1)(3x-1)=330.$$

is: (A): 0; (B): 1; (C): 2; (D): 3; (E): None of the above.

Answer: (B).

Multiply both sides of the equation by 2.3.4, we find

$$(12x-1)(12x-2)(12x-3)(12x-4)=11\times 10\times 9\times 8.$$

Left side is the product of 4 non-zero consecutive integers then all factors are the same sign. This argument follows that

$$\left[\begin{array}{c}(12x-1)(12x-2)(12x-3)(12x-4)=11\times 10\times 9\times 8\\ (12x-1)(12x-2)(12x-3)(12x-4)=(-11)\times(-10)\times(-9)\times(-8)\end{array}\right.$$

The 1st equation has a root $x=1$, the 2nd equation has no integer roots.

Question 6. Let ABC be a triangle with area 1 (cm^2). Points D, E and F lie on the sides AB, BC and CA, respectively. Prove that

$$\min\{\text{Area of }\Delta ADF, \text{Area of }\Delta BED, \text{Area of }\Delta CEF\}\le \frac{1}{4}\ (\text{cm}^2).$$

Answer.

From the equalities

$$\frac{S_{ADF}}{S_{ABC}} = \frac{AD \times AF}{AB \times AC}, \quad \frac{S_{BED}}{S_{ABC}} = \frac{BD \times BF}{AB \times AC}, \quad \frac{S_{CEF}}{S_{ABC}} = \frac{CE \times CF}{AB \times AC},$$

we find

$$\frac{S_{ADF} S_{BED} S_{CEF}}{(S_{ABC})^3} = \frac{(AD \times BD)(BE \times EC)(AF \times FC)}{AB^2 \times AC^2 \times BC^2}$$
$$\leq \frac{\frac{AD+DB}{2}\frac{BE+EC}{2}\frac{AF+FC}{2}}{AB^2 \times AC^2 \times BC^2} = \frac{1}{64}.$$

Hence,

$$S_{ADF} S_{BED} S_{CEF} \leq \frac{1}{64} = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}.$$

It follows that

$$\min\{\text{Area of } \Delta ADF, \text{Area of } \Delta BED, \text{Area of } \Delta CEF\} \leq \frac{1}{4} \ (\text{cm}^2).$$

Question 7. Let ABC be a triangle with $\widehat{A} = 90^0, \widehat{B} = 60^0$ and $BC = 1$cm. Draw outside of ΔABC three equilateral triangles ABD, ACE and BCF. Determine the area of ΔDEF.

Answer.
From the assumption, we get $AB = \frac{1}{2}$, $AC = \frac{\sqrt{3}}{2}$ and $\widehat{DBE} = 180^0$.

It is easy to check that

$$S_{ABD} = \frac{1}{2} S_{ABC} = \frac{\sqrt{3}}{16}, \quad S_{BCF} = 2S_{ABC} = \frac{\sqrt{3}}{16}.$$

Hence, $S_{DEF} = \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} = \frac{9\sqrt{3}}{16}\text{cm}^2$.

Question 8. Let $ABCDE$ be a convex pentagon. Given that

$$S_{\Delta ABC} = S_{\Delta BCD} = S_{\Delta CDE} = S_{\Delta DEA} = S_{\Delta EAB} = 2\text{cm}^2,$$

Find the area of the pentagon.

Answer.
From the assumption

$$S_{\Delta ABC} = S_{\Delta BCD} = S_{\Delta CDE} = S_{\Delta DEA} = S_{\Delta EAB} = 2\text{cm}^2,$$

we find $AB \parallel EC$, $BC \parallel AD$, $AC \parallel DE$, $AE \parallel BD$.

Let O be the common point of BD and CE. Denote $S_{BCO} = x$. Since $ABOE$ is a parallelogram, then $S_{ABE} = S_{BOE} = 2$ and

$$S_{ABCDE} = S_{ABE} + S_{BOE} + S_{CDE} + S_{BOC} = 6 + x.$$

From

$$\frac{S_{BOC}}{S_{DOC}} = \frac{BO}{OD} = \frac{S_{BOE}}{S_{DOE}}$$

it follows $\frac{x}{2-x} = \frac{2}{x}$ since $S_{BOC} = S_{DOE}$, i.e. $x^2 + 2x + 1 = 5$ and then $x = \sqrt{5} - 1$. Hence

$$S_{ABCDE} = S_{ABE} + S_{BOE} + S_{CDE} + S_{BOC} = 6 + x = 6 + \sqrt{5} - 1 = 5 + \sqrt{5}\text{cm}^2.$$

Question 9. Solve the following system in positive numbers

$$\begin{cases} x + y \leq 1 \\ \dfrac{2}{xy} + \dfrac{1}{x^2 + y^2} = 10. \end{cases}$$

Answer.

For every root (x, y) of the system, we find

$$10 = \frac{2}{xy} + \frac{1}{x^2+y^2} = \left(\frac{1}{2xy} + \frac{1}{x^2+y^2}\right) + \frac{6}{4xy} \geq \frac{4}{(x+y)^2} + \frac{6}{(x+y)^2} \geq 4 + 6 = 10.$$

Hence, the system is equivalent to

$$\begin{cases} x + y = 1 \\ x = y \end{cases} \Leftrightarrow (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Question 10. Consider the set of all rectangles with a given perimeter p. Find the largest value of

$$M = \frac{S}{2S + p + 2},$$

where S is denoted the area of the rectangle.

Answer.

Let a, b be the lengths of sides of the rectangle, then $2(a + b) = p$, $ab = S$. By the Cauchy inequality, $p = 2(a + b) \geq 2 \times 2\sqrt{ab} = 4\sqrt{S}$. It follows $S \leq \dfrac{p^2}{16}$. Note that $0 < M < 1$, then

$$M = \frac{S}{2S + p + 2} \leq \frac{\frac{p^2}{16}}{\frac{p^2}{16} + p + 2} = \frac{p^2}{p^2 + 16p + 32}.$$

The equality holds for $a = b$, i.e. $ABCD$ is a square.

Question 11. The positive numbers a, b, c, d, e are such that the following identity hold for all real number x.

$$(x+a)(x+b)(x+c) = x^3 + 3dx^2 + 3x + e^3.$$

Find the smallest value of d.

Answer.
From the identity

$$(x+a)(x+b)(x+c) = x^3 + 3dx^2 + 3x + e^3$$

we find

$$\begin{cases} d = \dfrac{a+b+c}{3} \\ ab + bc + ca = 3 \end{cases}$$

Hence, by Cauchy inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get $d = \dfrac{a+b+c}{3} \geq \sqrt{\dfrac{ab+bc+ca}{3}} = 1$. The equality holds for $a = b = c = 1$.

Question 12. If $f(x) = ax^2 + bx + c$ safisfies the condition

$$|f(x)| < 1, \ \ \forall x \in [-1, 1],$$

prove that the equation $f(x) = 2x^2 - 1$ has two real roots.

Answer.
Rewrite the equation $f(x) = 2x^2 - 1$ in the form

$$g(x) := (2-a)x^2 - bx - 1 - c = 0. \tag{1}$$

By the assumption,

$$\begin{cases} f(-1) = a - b + c \\ f(1) = a + b + c \\ f(0) = c \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{1}{2}[f(1) + f(-1)] - f(0) \\ b = \dfrac{1}{2}[f(1) - f(-1)] \\ c = f(0) \end{cases}$$

Hence, $|a| < 2$ and $|c| < 1$. These follow the equation (1) is a quadratic equation with $2 - a > 0$ and $-1 - c < 0$ then its discriminant $\Delta = b^2 - 4(2-a)(-1-c) > 0$, i.e. the equation (1) has real roots.

Question 13. Solve the system of equations

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6} \\ \dfrac{3}{x} + \dfrac{2}{y} = \dfrac{5}{6} \end{cases}$$

Answer.

It is easy to check that

$$\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{6} \\ \frac{3}{x}+\frac{2}{y}=\frac{5}{6} \end{cases} \Leftrightarrow \begin{cases} \frac{3}{x}+\frac{3}{y}=\frac{3}{6} \\ \frac{3}{x}+\frac{2}{y}=\frac{5}{6} \end{cases} \Leftrightarrow \begin{cases} \frac{1}{y}=-\frac{1}{3} \\ \frac{1}{x}=\frac{1}{2} \end{cases} \Leftrightarrow (x,y)=(2,-3).$$

Question 14. Solve the system of equations

$$\begin{cases} x^3+y &= x^2+1 \\ 2y^3+z &= 2y^2+1 \\ 3z^3+x &= 3z^2+1 \end{cases}$$

Answer.

Rewrite the system in the form

$$\begin{cases} x^2(x-1) &= 1-y \\ 2y^2(y-1) &= 1-z \\ 3z^2(z-1) &= 1-x \end{cases}$$

It follows that

$$(x-1)(y-1)(z-1)(6x^2y^2z^2+1)=0. \qquad (1)$$

Since $6x^2y^2z^2+1>0$ for all x,y,z then (1) $\Leftrightarrow x=1$ or $y=1$ or $z=1$. For all cases, we always obtain the unique solution $(x,y,z)=(1,1,1)$.

Question 15. Denote by $\mathbb{Q}$ and $\mathbb{N}^*$ the set of all rational and positive integer numbers, respectively. Suppose that $\frac{ax+b}{x}\in\mathbb{Q}$ for every $x\in\mathbb{N}^*$. Prove that there exist integers A,B,C such that

$$\frac{ax+b}{x}=\frac{Ax+B}{Cx} \text{ for all } x\in\mathbb{N}^*.$$

Answer.

Putting $x=1, x=2$ in $\frac{ax+b}{x}$ we get $a+b=p$, $\frac{2a+b}{2}=q$, where $p,q\in\mathbb{Q}$. So $a=2q-p\in\mathbb{Q}$ and $b=2p-2q\in\mathbb{Q}$. Write $a=\frac{M}{N}$, $b=\frac{P}{Q}$, where M,N,P,Q are integers. Hence

$$\frac{ax+b}{x}=\frac{\frac{M}{N}x+\frac{P}{Q}}{x}=\frac{(MQ)x+(PN)}{(NQ)x},$$

which was to be proved.