Hanoi Mathematical Society

Hanoi Open Mathematics Competition 2013

Senior Section

Sunday, 24 March 2013 08h30-11h30

Question 1. How may three-digit perfect squares are there such that if each digit is increased by one, the resulting number is also a perfect square?

(A): 1; (B): 2; (C): 4; (D): 8; (E) None of the above.

Answer: (A-E).

Question 2. The smallest value of the function

$$f(x) = |x| + \left|\frac{1 - 2013x}{2013 - x}\right|$$

where $x \in [-1, 1]$ is

(A): $\frac{1}{2012}$; (B): $\frac{1}{2013}$; (C): $\frac{1}{2014}$; (D): $\frac{1}{2015}$; (E): None of the above.

Answer: (B).

Note that $f(x) = |x| + \left|\frac{\frac{1}{2013} - x}{1 - \frac{x}{2013}}\right| \geq\geq |x| + \frac{1}{2013} - |x| = \frac{1}{2013}$. And $f(x) = \frac{1}{2013}$ for $x = \frac{1}{2013}$.

Question 3. What is the largest integer not exceeding $8x^3+6x-1$, where $x = \frac{1}{2}\left(\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}\right)$?

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Answer: (C).

Note that $4x^3 + 3x = 2$, then $8x^3 + 6x - 1 = 2.2 - 1 = 3$.

Question 4. Let $x_0 = [\alpha]$, $x_1 = [2\alpha] - [\alpha]$, $x_2 = [3\alpha] - [2\alpha]$, $x_4 = [5\alpha] - [4\alpha]$, $x_5 = [6\alpha] - [5\alpha]$, $\ldots$, where $\alpha = \frac{\sqrt{2013}}{\sqrt{2014}}$. The value of x_9 is

(A): 2; (B): 3; (C): 4; (D): 5; (E): None of the above.

Answer: (E).

Note that $[(n+1)\alpha] \leq [n\alpha + 1] = [n\alpha] + 1$ for all $n \in \mathbb{N}$. Hence $[(n+1)\alpha] - [n\alpha] \leq 1$ for all $n \in \mathbb{N}$.

Question 5. The number n is called a composite number if it can be written in the form $n = a \times b$, where a, b are positive integers greater than 1.

Write number 2013 in a sum of m composite numbers. What is the largest value of m?

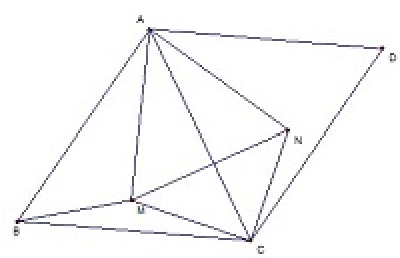
(A): 500; (B): 501; (C): 502; (D): 503; (E): None of the above.

Answer: (C).

Question 6. Let be given $a \in \{0, 1, 2, 3, \ldots, 1006\}$. Find all $n \in \{0, 1, 2, 3, \ldots, 2013\}$ such that $C_{2013}^{n} > C_{2013}^{a}$, where $C_m^k = \dfrac{m!}{k!(m-k)!}$.

Answer.
If $a = 1006$ there is no n.
If $a \in \{0, 1, 2, 3, \ldots, 1005\}$, then $x \in \{a+1, a+2, \ldots, 2012 - a\}$.

Question 7. Let ABC be an equilateral triangle and a point M inside the triangle such that $MA^2 = MB^2 + MC^2$. Draw an equilateral triangle ACD where $D \neq B$. Let the point N inside ΔACD such that AMN is an equilateral triangle. Determine $\widehat{BMC}$.



Answer.
Putting $MA = a; MB = b; MC = c.$ then $a^2 = b^2 + c^2$. It is easy to check that $\Delta ANC = \Delta AMB$ then $NC = MB = b$.

In $\triangle\ MCN$ we find $NC^2 + MC^2 = b^2 + c^2 = a^2 = MN^2$, so $\widehat{MCN} = 90^0$. Hence

$$\widehat{MBC} + \widehat{MCB} = \widehat{NCD} + \widehat{MCB} = 120^0 - 90^0 = 30^0$$

It follows $\widehat{BMC} = 150^0$.

Question 8. Let $ABCDE$ be a convex pentagon and

$$\text{area of } \Delta ABC = \text{area of } \Delta BCD = \text{area of } \Delta CDE = \text{area of } \Delta DEA = \text{area of } \Delta EAB.$$

Given that area of $\Delta ABCDE = 2$. Evaluate the area of area of ΔABC.

Answer.
Write $S_{ABC} :=$ Area of ABC. Since $S_{ABC} = S_{ABE}$ $(= a)$ then CE is parallel to AB. $S_{DBC} = S_{ECD}$ follows $BE \parallel CD$. Similarly, we find $AC \parallel DE$; $BD \parallel AE$; $AD \parallel BC$.

Let O be the common point of BD and CE and $S_{BCO} = x$. Since $ABOE$ is a parallelogram then $S_{ABE} = S_{BOE} = a$. Hence $2 = S_{ABCDE} = S_{ABE} + S_{BOE} + S_{CDE} + S_{BOC} = 3a + x$. Hence $a = \dfrac{2-x}{3}$.

Note that $\dfrac{S_{BOC}}{S_{DOC}} = \dfrac{BO}{OD} = \dfrac{S_{BOE}}{S_{DOE}}$ then we have $5x^2 - 10x + 4 = 0$. It follows $x = \dfrac{5-\sqrt{5}}{5}$.

Question 9. A given polynomial $P(t) = t^3 + at^2 + bt + c$ has 3 distinct real roots. If the equation $(x^2+x+2013)^3 + a(x^2+x+2013)^2 + b(x^2+x+2013) + c = 0$ has no real roots, prove that $P(2013) > \dfrac{1}{64}$.

Answer.
By condition,

$$P(x) = (x-x_1)(x-x_2)(x-x_3),$$

Then

$$P(Q(x)) = (Q(x)-x_1)(Q(x)-x_2)(Q(x)-x_3),$$

where

$$Q(x) - x_i \neq 0, i = 1, 2, 3.$$

Calculating the discriminants, we get

$$D_i = 1 - 4(2013 - x_i) < 0$$

and

$$2013 - x_i > \frac{1}{4}.$$

Therefore

$$P(2013) = (2013 - x_i)(2013 - x_2)(2013 - x_3) > \frac{1}{4}.\frac{1}{4}.\frac{1}{4} = \frac{1}{64}.$$

Question 10. Consider the set of all rectangles with a given area S. Find the largest value of

$$M = \frac{16-p}{p^2+2p},$$

where p is the perimeter of the rectangle.

Answer.
Let a, b be the side-lengths of the rectangle. Consider the case of all rectangles such that there esists a $p = 2(a+b)$ possessing the property $0 < p < 16$ with a given area S. Using the inequality $p = 2(a+b) \geq 4\sqrt{S}$, we find

$$M = \frac{16-p}{p^2+2p} \leq \frac{16-4\sqrt{S}}{(4\sqrt{S})^2 + 2 \times 4\sqrt{S}} = \frac{4-\sqrt{S}}{4S+2\sqrt{S}}.$$

The equality holds iff $a = b$, i.e. $ABCD$ is a square.

Note that, if for all side-lengths of rectangles $p > 16$, then $M(p) = \dfrac{16-p}{p^2+2p} < 0$ and for that cases, there is no largest value of M. Indeed, it is easy to see that for every $M_0 < 0$, always there exists p such that $M(p) = \dfrac{16-p}{p^2+2p} > M_0$.

Question 11. The positive numbers a, b, c, d, p, q are such that

$$(x+a)(x+b)(x+c)(x+d) = x^4 + 4px^3 + 6x^2 + 4qx + 1 \text{ holds for all real numbers } x.$$

Find the smallest value of p or the largest value of q.

Answer.
The identity

$$(x+a)(x+b)(x+c)(x+d) = x^4 + 4px^3 + 6x^2 + 4qx + 1 \text{ holds for all real numbers } x$$

means that $p = \dfrac{a+b+c+d}{4}$, $abcd = 1$. The AM-GM inequality gives $p \geq 1$ and $\min p = \dfrac{1}{4}$ when $a = b = c = d = 1$. On the other hand,

$$\frac{ab+ac+ad+bc+bd+cd}{6} = 1$$

and

$$\frac{abc+abd+acd+bcd}{4} = q.$$

From the inequality

$$\Big(\frac{ab+ac+ad+bc+bd+cd}{6}\Big)^3 \geq \Big(\frac{abc+abd+acd+bcd}{4}\Big)^2,$$

we find $q \leq 1$ and $\max q = 1$ when $a = b = c = d = 1$.

Question 12. The function $f(x) = ax^2 + bx + c$ safisfies the following conditions: $f(\sqrt{2}) = 3$, and

$$|f(x)| \leq 1, \text{ for all } x \in [-1, 1].$$

Evaluate the value of $f(\sqrt{2013})$.

Answer.
Condider the polynomial $Q(x) := 2x^2 - 1 - f(x)$. Note that $\deg Q(x) \leq 2$ and $Q(-1) = 1 - f(-1) \geq 0$, $Q(0) = 1 - f(0) \leq 0$, $Q(1) = 1 - f(1) \geq 0$. Then the equation $Q(x) = 0$ has at least 2 real roots in $[-1, 1]$. On the other hand, $Q(\sqrt{2}) = f(\sqrt{2}) - 2(\sqrt{2})^2 - 1 = 0$. So $Q(x) \equiv 0$. It means that $f(x) = 2x^2 - 1$ and $f(\sqrt{2013}) = 2.2013 - 1 = 4025$.

Question 13. Solve the system of equations

$$\begin{cases} xy = 1 \\ \dfrac{x}{x^4+y^2} + \dfrac{y}{x^2+y^4} = 1 \end{cases}$$

Answer.

Using the inequalities $\dfrac{x^2y}{x^4+y^2} = \dfrac{1}{\frac{x^2}{y}+\frac{y}{x^2}} \le \dfrac{1}{2}$ and $\dfrac{xy^2}{x^2+y^4} = \dfrac{1}{\frac{x}{y^2}+\frac{y^2}{x}} \le \dfrac{1}{2}$. Hence, for every root (x, y) of the system, we find

$$\begin{cases} xy = 1 \\ 1 = \dfrac{x}{x^4+y^2} + \dfrac{y}{x^2+y^4} = \dfrac{x^2y}{x^4+y^2} + \dfrac{xy^2}{x^2+y^4} \le 1. \end{cases}$$

So x, y must satisfy the system

$$\begin{cases} xy = 1 \\ \frac{x}{y^2} = \frac{y^2}{x} \\ \frac{x^2}{y} = \frac{y}{x^2} \end{cases}$$

and then $x = y = 1$. Indeed, $(x, y) = (1, 1)$ satisfies the given system.

Question 14. Solve the system of equations:

$$\begin{cases} x^3 + \dfrac{1}{3}y & = x^2 + x - \dfrac{4}{3} \\ y^3 + \dfrac{1}{4}z & = y^2 + y - \dfrac{5}{4} \\ z^3 + \dfrac{1}{5}x & = z^2 + z - \dfrac{6}{5} \end{cases}$$

Answer.

Rewrite the system in the form

$$\begin{cases} (x+1)(x-1)^2 = -\dfrac{1}{3}(y+1) \\ (y+1)(y-1)^2 = -\dfrac{1}{3}(z+1) \\ (z+1)(z-1)^2 = -\dfrac{1}{3}(x+1) \end{cases}$$

We find $(x+1)(y+1)(z+1)[(x-1)^2(y-1)^2(y-1)^2 + \dfrac{1}{3}] = 0$. It follows $(x, y, z) = (-1, -1, -1)$ is a unique solution of the given system.

Question 15. Denote by $\mathbb{Q}$ and $\mathbb{N}^*$ the set of all rational and positive integral numbers, respectively. Suppose that $\dfrac{ax+b}{cx+d} \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. Prove that there exist integers A, B, C, D such that

$$\frac{ax+b}{cx+d} = \frac{Ax+B}{Cx+D} \text{ for all } x \in \mathbb{N}^*.$$

Answer.

Write $f(x) = \dfrac{ax+b}{cx+d}$.

- If $c = 0$ then $f(x) = \alpha x + \beta \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. It follows $\alpha, \beta \in \mathbb{Q}$ and $\alpha = \dfrac{P}{Q}$, $\beta = \dfrac{M}{N}$, where $P, Q, M, N \in \mathbb{Z}$. Thus, $f(x) = \dfrac{P}{Q}x + \dfrac{M}{N} = \dfrac{PNx + QN}{0x + QN}$.

- If $a = 0$, $b = 0$ then $f(x) \equiv 0$.

- If $a = 0$, $b \neq 0$ then $f(x) = \dfrac{1}{\alpha x + \beta} \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. It follows $\alpha, \beta \in \mathbb{Q}$ and $\alpha = \dfrac{P}{Q}$, $\beta = \dfrac{M}{N}$ then $f(x) = \dfrac{0x + QN}{PNx + QN}$.

- If $ad - bc = 0$ then $f(x) \equiv c \in \mathbb{Q}$.

- Now consider the case $a \neq 0, c \neq 0, ad - bc \neq 0$. By the assumption, $f(1) = \dfrac{P_1}{Q_1}$, $P_1, Q_1 \in \mathbb{Z}$. Write

$$f(x) - f(1) = \frac{x - 1}{\gamma x + \delta}, \tag{$*$}$$

then $\gamma, \delta \in \mathbb{Q}$ and $\gamma = \dfrac{M_1}{N_1}$, $\delta = \dfrac{M_2}{N_2}$, where $M_1, M_2, N_1, N_2 \in \mathbb{Z}$. Putting the values of $f(1), \gamma, \delta$ in the formula $(*)$, we find

$$\frac{ax + b}{cx + d} = \frac{Ax + B}{Cx + D} \text{ for all } x \in \mathbb{N}^*.$$
