International Mathematical Competition 2010
Blagoevgrad, Bulgaria
First day
Time allowed: 5 hours

Problem 1 (10 points). Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}\mathrm{d}x \ge e^{-a^2} - e^{-b^2}.$$

Solution 1. Let $f(x) = \int\limits_a^x (t^2+1)e^{-t^2}\mathrm{d}t$ and let $g(x) = -e^{-x^2}$; both functions are increasing. By Cauchy's Mean Value Theorem, there exists a real number $x_0 \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)} = \frac{(x_0^2+1)e^{-x_0^2}}{2x_0 e^{-x_0^2}} = \frac{1}{2}\Big(x_0 + \frac{1}{x_0}\Big) \ge 1.$$

Then

$$\int_a^b (x^2+1)e^{-x^2}\mathrm{d}x = f(b) - f(a) \ge g(b) - g(a) = e^{-a^2} - e^{-b^2}.$$

Solution 2. The inequality $x^1 + 1 \ge 2x$ follows

$$\int_a^b (x^2+1)e^{-x^2}\mathrm{d}x \ge \int_a^b 2xe^{-x^2}\mathrm{d}x = -e^{-x^2}\Big|_a^b = e^{-a^2} - e^{-b^2}.$$

Problem 2 (10 points). Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots$$

Solution 1. Let

$$\sum_{k=0}^{\infty} \frac{x^{4k+4}}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

The power series converges for $|x| \le 1$ and our goal is to compute $F(1)$.

Differentiating 4 times, we get

$$F^{(4)}(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

Since $F(0) = F'(0) = F''(0) = F"(0) = 0$ anf F is continous at $1-0$ by Abel's continuity theorem, integrating 4 times we get

$$F'''(y) = F'''(0) + \int_0^y F^{(4)}(x)\mathrm{d}x = \int_0^y \frac{\mathrm{d}x}{1-x^4} = \frac{1}{2}\arctan y + \frac{1}{4}\log(1+y) - \frac{1}{4}\log(1-y),$$

$$F''(z) = F''(0) + \int_0^z F^{(3)}(y)\mathrm{d}y = \int_0^z \Big(\frac{1}{2}\arctan y + \frac{1}{4}\log(1+y) - \frac{1}{4}\log(1-y)\Big)\mathrm{d}y =$$

$$\frac{1}{2}\Big(z\arctan z - \int_0^z \frac{y\mathrm{d}y}{1+y^2}\Big) + \frac{1}{4}\Big((1+z)\log(1+z) - \int_0^z \mathrm{d}y\Big) + \frac{1}{4}\Big((1-z)\log(1-z) + \int_0^z \mathrm{d}y\Big)$$

$$= \frac{1}{2} z \arctan z - \frac{1}{4} \log(1+z^2) + \frac{1}{4}(1+z)\log(1+z) + \frac{1}{4}(1-z)\log(1-z),$$

$$F'(t) = \int_0^t \Big(\frac{1}{2} z \arctan z - \frac{1}{4} \log(1+z^2) + \frac{1}{4}(1+z)\log(1+z) + \frac{1}{4}(1-z)\log(1-z) \Big) \mathrm{d}z =$$

$$= \frac{1}{4}\Big((1+t^2)\arctan t - t\Big) - \frac{1}{4}\Big(t\log(1+t^2) - 2t + 2\arctan t\Big) +$$

$$+ \frac{1}{8}\Big((1+t^2)\log(1+t) - t - \frac{1}{2}t^2\Big) - \frac{1}{8}\Big((1-t^2)\log(1-t) + t - \frac{1}{2}t^2\Big) =$$

$$= \frac{1}{4}(-1+t^2)\arctan t - \frac{1}{4} t\log(1+t^2) + \frac{1}{8}(1+t^2)\log(1+t) - \frac{1}{8}(1-t^2)\log(1-t),$$

$$F(1) = \int_0^1 F'(t)\mathrm{d}t =$$

$$= \int_0^1 \Big(\frac{1}{4}(-1+t^2)\arctan t - \frac{1}{4} t\log(1+t^2) + \frac{1}{8}(1+t^2)\log(1+t) - \frac{1}{8}(1-t^2)\log(1-t) \Big) \mathrm{d}t$$

$$= \Big[\frac{-3t+t^3}{12} \arctan t + \frac{1-3t^2}{24}\log(1+t^2) + \frac{(1+t)^3}{24}\log(1-t) \Big]_0^1 = \frac{\ln 2}{4} - \frac{\pi}{24}.$$

Remark. The computation can be shorter if we change the order of integrations

$$F(1) = \int_{t=0}^1 \int_{z=0}^t \int_{y=0}^z \int_{x=0}^y \frac{1}{1-x^4} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t = \int_{x=0}^1 \frac{1}{1-x^4} \int_{y=x}^1 \int_{z=y}^1 \int_{t=z}^1 \mathrm{d}t\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

$$= \int_{x=0}^1 \frac{1}{1-x^4} \Big(\frac{1}{6} \int_{y=x}^1 \int_{z=y}^1 \int_{t=z}^1 \mathrm{d}t\mathrm{d}z\mathrm{d}y \Big) \mathrm{d}x = \int_0^1 \frac{1}{1-x^4} \cdot \frac{(1-x)^3}{6} \mathrm{d}x$$

$$= \Big[-\frac{1}{6}\arctan x - \frac{1}{12}\log(1+x^2) + \frac{1}{3}\log(1+x) \Big]_0^1 = \frac{\ln 2}{4} - \frac{\pi}{24}.$$

Solution 2. Let

$$A_m = \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} =$$

$$= \sum_{k=0}^{\infty} \Big(\frac{1}{6}\cdot\frac{1}{4k+1} - \frac{1}{2}\cdot\frac{1}{4k+2} + \frac{1}{2}\cdot\frac{1}{4k+3} - \frac{1}{6}\cdot\frac{1}{4k+4} \Big) = \frac{1}{3}C_m - \frac{1}{6}B_m - \frac{1}{6}D_m,$$

where

$$C_m = \sum_{k=0}^{\infty} \Big(\frac{1}{4k+1} - \frac{1}{4k+2} + \frac{1}{4k+3} - \frac{1}{4k+4} \Big),$$

$$B_m = \sum_{k=0}^{\infty} \Big(\frac{1}{4k+1} - \frac{1}{4k+3} \Big), \quad D_m = \sum_{k=0}^{\infty} \Big(\frac{1}{4k+2} - \frac{1}{4k+4} \Big).$$

Therefore,

$$\lim_{m\to\infty} A_m = \lim_{m\to\infty} \frac{2C_m - B_m - D_m}{6} = \frac{2\ln 2 - \frac{\pi}{4} - \frac{1}{2}\ln 2}{6} = \frac{\ln 2}{4} - \frac{\pi}{24}.$$

Problem 3 (10 points). Define the sequence $x_1, x_2, \ldots,$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 x_2 \cdots x_n}{x_{n+1}}.$$

Solution. Let $y_n = n^2$. Then $y_{n+1} = (y_n - 2)^2$ and $y_{n+1} - 4 = y_n(y_n - 4)$. Since $y_2 = 9 > 5$, we have $y_3 = (y_2 - 2)^2 > 5$ and inductively $y_n > 5$, $n \geq 2$. Hence, $y_{n+1} - y_n = y_n^2 - 5y_n + 4 > 4$ for all $n \geq 2$, so $y_n \to \infty$.

By $y_{n+1} - 4 = y_n(y_n - 4)$, we find

$$\left(\frac{x_1 x_2 \cdots x_n}{x_{n+1}}\right)^2 = \frac{y_1 y_2 \cdots y_n}{y_{n+1}}$$

$$= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 y_2 \cdots y_n}{y_{n+1} - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 y_2 \cdots y_{n-1}}{y_n - 4} = \cdots$$

$$= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{1}{y_1 - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \to 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{x_1 x_2 \cdots x_n}{x_{n+1}} = 1.$$

Problem 4 (10 points). Let a, b be two integers and suppose that n is a positive integer for which the set

$$\mathbb{Z} \setminus \{ax^n + by^n \;\Big|\; x, y \in \mathbb{Z}\}$$

is finite. Prove that $n = 1$. Prove that $n = 1$.

Solution. Asumme that $n > 1$. Notice that n may be replaced by any prime divisor p of n. Moreover, a and b shoutd be coprime, otherwise the numbers not divisible by the greatest common divisor of a, b can not be represented as $ax^n + by^n$.

If $n = 2$, then the number of form $ax^n + by^n$ takes not all possible remanders modulo 8. If, say, b is even, then ax^2 takes at most three different remanders modulo 8. by^2 takes at most two, hence $ax^n + by^n$ takes at most $3 \times 3 = 6$ different remanders. If both a and b are odd, then $ax^n + by^n \equiv x^2 \pm y^2 (\mod 4)$: the expression $x^2 + y^2$ does not take the remanders 3 modulo 4 and $x^2 - y^2$ does take the remander 2 modulo 4.

Consider the case when $p \geq 3$. The pth powers take exactly p different remanders modulo p^2. Indeed, $(x + kp)^p$ and x^p have the same remander modulo p^2, and all numbers $0^p, 1^p, \ldots, (p-1)^p$ are different even modulo p. So, $ax^p + by^p$ take at most p^2 different remanders modulo p^2. If it takes strictly less then p^2 different remanders modulo p^2, we get infinitely many non-reprentable numbers. If it takes exactly p^2 remanders, then $ax^p + by^p$ is divisible by p^2, it is also divisible by p^p. Again we get infinitely many non-reprentable numbers, for example the numbers congruent to p^2 are non-reprentable numbers.

Problem 5 (10 points). Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.

Solution. The constraint can be written as

$$(a - bc)^2 \leq (1 - b^2)(1 - c^2). \tag{1}$$

By the Cauchy-Schwarz inequality,

$$(a^{n-1}+a^{n-2}bc+\cdots b^{n-1}c^{n-1})^2 \le (|a|^{n-1}+|a|^{n-2}|bc|+|bc|^{n-1})$$

$$\le (1+|bc|+\cdots+|bc|^{n-1})^2 \le (1+|b|^2+\cdots+|b|^{2(n-1)})\cdot(1+|c|^2+\cdots+|c|^{2(n-1)}).$$

Multilying by (1), we get

$$(a-bc)^2(a^{n-1}+a^{n-2}bc+\cdots b^{n-1}c^{n-1})^2 \le$$

$$(1-b^2)(1+|b|^2+\cdots+|b|^{2(n-1)})\cdot(1-c^2)(1+|c|^2+\cdots+|c|^{2(n-1)})$$

$$\Leftrightarrow (a^n-b^nc^n)^2 \le (1-b^{2n})(1-c^{2n}) \Leftrightarrow 1+2(abc)^n \ge a^{2n}+b^{2n}+c^{2n}.$$

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Second day

Time allowed: 5 hours

Problem 6 (10 points). (a) A sequence $x_1, x_2, \ldots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \quad \text{for all} \quad n \ge 1.$$

Does it follow that this sequence converges for all initial value x_1?

(b) A sequence $y_1, y_2, \ldots$ of real numbers satisfies

$$y_{n+1} = y_n \sin x_n \quad \text{for all} \quad n \ge 1.$$

Does it follow that this sequence converges for all initial value y_1?

Solution.

(a) NO. For example, for $x_1 = \pi$ we have $x_n = (-1)^n\pi$, and the sequence is divergent.

(b) YES. Notice that $|y_n|$ is nonincreasing and hence converges to some number $a \ge 0$.

If $a = 0$, then $\lim y_n = 0$ and we are done.

If $a > 0$, then $a = \lim|y_{n+1}| = \lim|y_n \sin y_n| = a|\sin a|$, so $\sin a = \pm 1$ and $a = \left(k+\frac{1}{2}\right)\pi$ for some nonnegative integer k.

Since the sequence $|y_n|$ is nonincreasing, there exists an index n_0 such that

$$\left(k+\frac{1}{2}\right)\pi \le |y_n| < (k+1)\pi \quad \text{for all} \quad n > n_0.$$

Then all the numbers $y_{n_0+1}, y_{n_0+2}, \ldots$ lie in the union of the intervals $\Big[\left(k+\frac{1}{2}\right)\pi, (k+1)\pi\Big)$ and $\Big(-(k+1)\pi, -\left(k+\frac{1}{2}\right)\pi\Big]$

Depending on the parity of k, in one of the intervals $\Big[\left(k+\frac{1}{2}\right)\pi, (k+1)\pi\Big)$ and $\Big(-(k+1)\pi, -\left(k+\frac{1}{2}\right)\pi\Big]$ the values of the sine function is positive, denote this interval by I_+. In the other interval the sin function is negative, denote this interval by I_-. If $y_n \in I_-$ for some $n > n_0$ then y_n and $y_{n+1} = y_n \sin y_n$ have opposite signs, so $y_{n+1} \in I_+$. On the other hand, if $y_n \in I_+$ for some $n > n_0$ then y_n and $y_{n+1} = y_n \sin y_n$ have the same sign, so $y_{n+1} \in I_+$. In both cases, $y_{n+1} \in I_+$.

We obtained that the numbers $y_{n_0+2}, y_{n_0+3}, \ldots$ lie in I_+, so they have the same sign. Since $|y_n|$ is convergent, this implies that the sequence $\{y_n\}$ is convergent as well.

Problem 7 (10 points). Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{n+1} - a_n \geq 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1 + \frac{1}{a_0}\Big(1 + \frac{1}{a_1 - a_0}\Big)\cdots\Big(1 + \frac{1}{a_n - a_0}\Big) \leq \Big(1 + \frac{1}{a_0}\Big)\Big(1 + \frac{1}{a_1}\Big)\cdots\Big(1 + \frac{1}{a_n}\Big).$$

Solution. Apply induction on n. Considering the empty product as 1, we have equality for $n = 0$.

Now assume that the statement is true for some n and prove it for $n+1$. For $n+1$, the staterment can be written as the sum of the inequalities

$$1 + \frac{1}{a_0}\Big(1 + \frac{1}{a_1 - a_0}\Big)\cdots\Big(1 + \frac{1}{a_n - a_0}\Big) \leq \Big(1 + \frac{1}{a_0}\Big)\Big(1 + \frac{1}{a_1}\Big)\cdots\Big(1 + \frac{1}{a_n}\Big)$$

(which is the induction hypothesis) and

$$\frac{1}{a_0}\Big(1+\frac{1}{a_1 - a_0}\Big)\cdots\Big(1+\frac{1}{a_n - a_0}\Big)\cdot\frac{1}{a_{n+1} - a_0} \leq \Big(1+\frac{1}{a_0}\Big)\Big(1+\frac{1}{a_1}\Big)\cdots\Big(1+\frac{1}{a_n}\Big)\cdot\frac{1}{a_{n+1}}. \quad (1)$$

Hence, to complete the solution it is sufficient to prove (1).

To prove (1), apply a second induction. For $n = 0$, we have to verify

$$\frac{1}{a_0}\cdot\frac{1}{a_1 - a_0} \leq \Big(1 + \frac{1}{a_0}\Big)\frac{1}{a_1}.$$

Multiplying by $a_0 a_1 (a_1 - a_0)$, this is equivalent with

$$a_1 \leq (a_0 + 1)(a_1 - a_0) \Leftrightarrow a_0 \leq a_0 a_1 - a_0^2 \Leftrightarrow 1 \leq a_1 - a_0.$$

For the induction step it is sufficient that

$$\Big(1 + \frac{1}{a_{n+1} - a_0}\Big)\cdot\frac{a_{n+1} - a_0}{a_{n+2} - a_0} \leq \Big(1 + \frac{1}{a_{n+1}}\Big)\cdot\frac{a_{n+1}}{a_{n+2}}.$$

Multiplying by $(a_{n+2} - a_0)a_{n+2}$,

$$(a_{n+1} - a_0 + 1)a_{n+2} \leq (a_{n+1} + 1)(a_{n+2} - a_0)$$

$$\Leftrightarrow a_0 \leq a_0 a_{n+2} - a_0 a_{n+1} \Leftrightarrow 1 \leq a_{n+2} - a_{n+1}.$$

Note that (from the solution) the equality holds if and only if $a_{k+1} - a_k = 1$ for all k.

Remark. The statement of the problem is a direct corollary of the identity

$$1 + \sum_{i=0}^{n}\Big(\frac{1}{a_i}\prod_{j\neq i}\big(1 + \frac{1}{a_j - a_i}\big)\Big) = \prod_{i=0}^{n}\Big(1 + \frac{1}{a_i}\Big).$$

Problem 8 (10 points). Denote by S_n the group of permutations of the sequence $(1, 2, \ldots, n)$. Suppose that G is a subgroup of S_n, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \ldots, n\}$ for which $\pi(k) = k$. (Here e is the unit element in the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.

Solution. Let us consider the action of G on the set $X = \{1, 2, \ldots, n\}$. Let

$$G_x = \{g \in G : \; g(x) = x\} \;\text{ and }\; Gx = \{g(x) : \; g \in G\}$$

be the stabilizer and the orbit of $x \in X$ under this action, respectively. The condition of the problem state that

$$G = \bigcup_{x \in X} G_x \tag{1}$$

and

$$G_x \cap G_y = \{e\} \text{ for all } x \neq y. \tag{2}$$

We need to prove that $G_x = G$ for some $x \in X$.

Let $Gx_1, Gx_2, \ldots, Gx_k$ be the distinct orbits of the action of G. Then one can write (1) as

$$G = \bigcup_{i=1}^{k} \bigcup_{y \in Gx_i} G_y. \tag{3}$$

It is well known that

$$|Gx| = \frac{|G|}{|G_x|}. \tag{4}$$

Also note that if $y \in Gx$ then $Gy = Gx$ and thus $|Gy| = |Gx|$. Therefore,

$$|G_x| = \frac{|G|}{|Gx|} = \frac{|G|}{|Gy|} = |G_y| \text{ for all } y \in Gx. \tag{5}$$

Combining (3),(2), (4) and (5) we gwt

$$|G| - 1 = |G \setminus \{e\}| = \Big| \bigcup_{i=1}^{k} \bigcup_{y \in Gx_i} G_y \setminus \{e\} \Big| = \sum_{i=1}^{k} \frac{|G|}{|G_{x_i}|}(|G_{x_i}| - 1),$$

hence

$$1 - \frac{1}{|G|} = \sum_{i=1}^{k} \Big(1 - \frac{1}{|G_{x_i}|}\Big). \tag{6}$$

If for some $i, j \in \{1, 2, \ldots, k\}$ we have $|G_{x_i}|, |G_{x_j}| \geq 2$ then

$$\sum_{i=1}^{k} \Big(1 - \frac{1}{|G_{x_i}|}\Big) \geq \Big(1 - \frac{1}{2}\Big) + \Big(1 - \frac{1}{2}\Big) = 1 > 1 - \frac{1}{|G|},$$

which contradics with (6), thus we can assume that

$$|G_{x_1}| = |G_{x_2}| = \cdots = |G_{x_{k-1}}| = 1.$$

Then from (6) we get $|G_{x_k}| = |G|$, hence$G_{x_k} = G$.

Problem 9 (10 points). Let A be symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of matrix A^n has a zero entry.

Solution. Denote by e_k ($1 \leq k \leq m$) the m-dimensional vector over F_2, whose k-th entry is 1 and all the other elements are 0. Furthermore, let u be the vector whose all entries are 1. The k-th column of A^n is $A^n e_k$. So the statement can be written as $A^n e_k \neq u$ for all $1 \leq k \leq m$ and all $n \geq 1$.

For every pair of vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define the bilinear form $(x, y) = x^T y = x_1 y_1 + \cdots + x_m y_m$. The product (x, y) has all basic properties of scalar product (except the property that $(x, x) = 0$ implies $x = 0$). Moreover, we have $(x, x) = (x, u)$ for every vector $x \in F_2^m$.

It is also easy to check that $(w, Aw) = w^T A w = 0$ for all vectors w, since A is symmetric and its diagonal elements are 0.

Lemma. Suppose that $v \in F_2^m$ is a vector such that $A^m v = u$ for some $n \geq 1$. Then $(v, v) = 0$.

Proof. Apply induction by n. For odd values of n we prove the lemma directly. Let $n = 2k + 1$ and $w = A^k v$, then

$$(v, v) = (v, u) = (v, A^n v) = v^T A^n v = v^T A^{2k+1} v = (A^k v, A^{k+1} v) = (w, Aw) = 0.$$

Now suppose that n is even, $n = 2k$, and the lemma is true for all smaller values of n. Let $w = A^k v$, then $A^k w = A^n v = u$ and thus we have $(w, w) = 0$ by the induction hypothesis. Hence,

$$(v, v) = (v, u) = (v, A^n v) = v^T A^{2k} v = (A^k v)^T (A^k v) = (A^k v, A^k v) = (w, w) = 0.$$

The lemma is proved.

Now suppose that $A^n e_k = u$ for some $1 \leq k \leq m$ and positive n. By the lemma, we should have $(e_k, e_k) = 0$. But this is imposible because $(e_k, e_k) = 1 \neq 0$.

Problem 10 (10 points). Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\Big(y + \frac{k}{p}\Big) = 0$$

for every prime number p and every real number y.

Solution. Let $N > 1$ be some integer to be defined later, and consider set of all real polynomials

$$\mathcal{J}_N = \Big\{ c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}[x] \;\Big|\; \forall x \in \mathbb{R} \; \sum_{k=0}^{n} c_k f\Big(x + \frac{k}{N}\Big) = 0 \Big\}.$$

Notice that $0 \in \mathcal{J}_N$, any linear combinations of any elements in $\mathcal{J}_N$ is in $\mathcal{J}_N$, and for every $P(x) \in \mathcal{J}_N$ we have $xP(x) \in \mathcal{J}_N$. Hence, $\mathcal{J}_N$ is an ideal of the ring $\mathbb{R}[x]$.

By the problem's conditions, for every prime divisors of N we have $\dfrac{x^N - 1}{x^{N/p} - 1} \in \mathcal{J}_N$. Since $\mathbb{R}[x]$ is a principle ideal domain (due to the Euclidean algorithm), the greatest common divisor is the intersection of such sets: it can be seen that the intersection consist of the primitive Nth roots of unity. Therefore,

$$\gcd \Big\{ \frac{x^N - 1}{x^{N/p} - 1} \mid p|N \Big\} = \Phi_N(x)$$

is the Nth cyclotomic polynomial. So $\Phi_N \in \mathcal{J}_N$, which polynomial has degree $\varphi(N)$.

Now we choose N in such a way that $\dfrac{\varphi(N)}{N} < b - a$. It is well known that $\liminf\limits_{N \to \infty} \dfrac{\varphi(N)}{N} = 0$, so there exists such a value for N. Let $\Phi_N(x) = a_0 + a_1 x + \cdots + a_{\varphi(N)} x^{\varphi(N)}$ where $a_\varphi = 1$ and $|a_0| = 1$.

Then, by the definition of $\mathcal{J}_N$, we have

$$\sum_{k=0}^{\varphi(N)} a_k f\Big(x - \frac{\varphi(N) - k}{N}\Big) = 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

If $x \in \Big[b, b + \dfrac{1}{N}\Big)$, then

$$f(x) = -\sum_{k=0}^{\varphi(N)} a_k f\Big(x - \frac{\varphi(N) - k}{N}\Big).$$

On the right-hand side, all numbers $x - \dfrac{\varphi(N) - k}{N}$ lie in (a, b). Therefore the right-hand side is zero and $f(x) = 0$ for all $x \in \left[b, b + \dfrac{1}{N}\right)$. It can be obtained similarly that $f(x) = 0$ for all $x \in \left(a - \dfrac{1}{N}, a\right]$ as well. Hence, $f = 0$ in the interval $\left(a - \dfrac{1}{N}, b + \dfrac{1}{N}\right)$. Continuing in this fashion we see that f must vanish everywhere.