International Mathematics Competition for University Students

July 25-30 2009, Budapest, Hungary

Day 1

Problem 1. Suppose that f and g are real-valued functions on the real line and $f(r) \leq g(r)$ for every rational r. Does this imply that $f(x) \leq g(x)$ for every real x if

a) f and g are non-decreasing?

b) f and g are continuous?

Problem 2. Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A - B)C = BA^{-1}$, then $C(A - B) = A^{-1}B$.

Problem 3. In a town every two residents who are not friends have at least a friend in common, and no one is a fiend, of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

Problem 4. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \cdots c_n \geq 0$ is a sequence of real numbers which is convex (i.e. $2c_k \leq c_{k-1} + c_{k+1}$ for every $k =$

$1, 2, \ldots, n-1$), and consider the polynomials

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n.$$

Prove that

$$\max_{|z| \le 1} |q(z)| \le \max_{|z| \le 1} |p(z)|.$$

Problem 5. Let n be a positive integer. An n-simplex in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its vertices, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)C(S).$$

Day 2

Problem 1. Let l be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to l is greater or equal to two times the distance between X and P. If the distance from P to l is $d > 0$, evaluate the volume of S.

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$

$$f''(x) - 5f'(x) + 6f(x) \ge 0.$$

Prove that for all $x \in [0, \infty)$

$$f(x) \ge 3e^{2x} - 2e^{3x}.$$

Problem 3. Let A and B be two complex square matrices such that

$$A^2 B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

Problem 4. Let p be a prime number and $\mathbb{F}_p$ be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in $\mathbb{F}_p$ such that

- the polynomial $x+1$ and $x^{p-2}+x^{p-3}+\cdots+x^2+2x+1$ are in W, and
- for any polynomials $h_1(x)$ and $h_2(x)$ in W the polynomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$, is also in W.

How many polynomials are there in W?

Problem 5. Let $\mathbb{M}$ be the vector space of $m \times p$ real matrices. For a vector subspace $S \subset \mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S.

Say that a vector subspace $T \subset \mathbb{M}$ is a convering matrix space if

$$\cup_{A \in T, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a T is minimal if it does not contain a proper vector subspace $S \subset T$ which is also a convering matrix space.

(a)(8 points) Let T be a minimal convering matrix space and let $n = \dim T$. Prove that

$$\delta(T) \leq \binom{n}{2}.$$

(b)(2 points) Prove that for every positive integer n we can find m and p, and a minimal convering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$.