INTERNATIONAL MATHEMATICS COMPETITIONS

FOR UNIVERSITY STUDENTS

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SELECTION OF PROBLEMS AND SOLUTIONS

Hanoi, 2009

Contents

Chapter 1

Questions

1.1 Olympic 1994

1.1.1 Day 1, 1994

Problem 1. (13 points)

a) Let A be a $n \times n, n \geq 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.

b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix}$$

Problem 2. (13 points)

Let $f \in \mathcal{C}^1(a, b)$, $\lim\limits_{x \to a+} f(x) = \infty$, $\lim\limits_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Problem 3. (13 points)

Give a set S of $2n - 1, n \in \mathbb{N}$, different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \cdots a_n > 0$ we have that $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, is an irrational number.

Problem 4. (18 points)

Let $\alpha \in \mathbb{R}\backslash\{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

b) Show that there exists $k \geq 1$ such that $F^k = 0$.

Problem 5. (18 points)

a) Let $f \in \mathcal{C}[0, b], g \in \mathcal{C}(\mathbb{R})$ and let g be periodic with period b. Prove that $\int\limits_0^b f(x)g(nx)dx$ has a limit as $n \to \infty$ and

$$\lim_{n\to\infty} \int\limits_0^b f(x)g(nx)dx = \frac{1}{b} \int\limits_0^b f(x)dx \int\limits_0^b g(x)dx.$$

b) Find

$$\lim_{n\to\infty} \int\limits_0^\pi \frac{\sin x}{1 + 3\cos^2 nx} dx.$$

Problem 6. (25 points)

Let $f \in \mathcal{C}^2[0, N]$ and $|f'(x)| < 1,\ f"(x) > 0$ for every $x \in [0, N]$. Let $0 \leq m_0 < m_1 < \cdots < m_k \leq N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0, 1, \ldots, k$. Denote bi = ni - ni-1 and ai = mi - mi-1 for i = 1,2, ... , k.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of $A > 1$ there are no more than $\dfrac{N}{A}$ indices j such that $a_j > A$.

c) Prove that $k \leq 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x), x \in [0, N]$).

1.1.2 Day 2, 1994

Problem 1. (14 points)

Let $f \in \mathcal{C}^1[a,b], f(a) = 0$ and suppose that $\lambda \in \mathbb{R}, \lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

Problem 2. (14 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

a) Prove that f attains its minimum and its maximum.

b) Determine all points (x,y) such that $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them f has global or local minimum or maximum.

Problem 3. (14 points)

Let f be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers $a, b, a < b$, such that

$$\ln \Big(\frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)} \Big) = b - a$$

there is a number c in the open interval (a,b) for which

$$f^{(n+1)}(c) = f(c).$$

Note that ln denotes the natural logarithm.

Problem 4. (18 points)

Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \dots (x - c_k)^{d_k},$$

where $c_1, c_2, \dots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diagonal, etc. and $d_1 + d_2 + \cdots + d_k = n$).

Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

Problem 5. (18 points)

Let $x_1, x_2, \ldots, x_k$ be vectors of m-dimensional Euclidian space, such that $x_1 + x_2 + \cdots + x_k = 0$. Show that there exists a permutation π of the integers $\{1, 2, \ldots, k\}$ such that

$$\parallel \sum_{i=1}^{n} x_{\pi(i)} \parallel \leq \left(\sum_{i=1}^{k} \parallel x_i \parallel^2 \right)^{1/2} \text{ for each } n = 1, 2, \ldots, k.$$

Note that $\parallel . \parallel$ denotes the Euclidian norm.

Problem 6. (22 points)

Find $\lim\limits_{N\to\infty} \dfrac{ln^2 N}{N} \sum\limits_{k=2}^{N-2} \dfrac{1}{\ln k . ln(N-k)}$. Note that ln denotes the natural logarithm.

1.2 Olympic 1995

1.2.1 Day 1, 1995

Problem 1. (10 points)

Let X be a nonsingular matrix with columns $X_l, X_2, \ldots, X_n$. Let Y be a matrix with columns $X_2, X_3, \ldots, X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0's for eigenvalues.

Problem 2. (15 points)

Let f be a continuous function on $[0, 1]$ such that for every $x \in [0, 1]$ we have $\int\limits_x^1 f(t)dt \geq \dfrac{1-x^2}{2}$. Show that $\int\limits_0^1 f^2(t)dt \geq \dfrac{1}{3}$.

Problem 3. (15 points)

Let f be twice continuously differentiable on $(0, +\infty)$ such that

$$\lim_{x\to 0+} f'(x) = -\infty$$

and

$$\lim_{x\to 0+} f"(x) = +\infty.$$

Show that

$$\lim_{x\to 0+} \frac{f(x)}{f'(x)} = 0.$$

Problem 4. (15 points)

Let $F : (1, \infty) \to \mathbb{R}$ be the function defined by

$$F(x) := \int_x^{x^2} \frac{dt}{lnt}.$$

Show that F is one-to-one (i.e. injective) and find the range (i.e. set of values) of F.

Problem 5. (20 points)

Let A and B be real $n \times n$ matrices. Assume that there exist $n + 1$ different real numbers $t_l, t_2, \ldots, t_{n+1}$ such that the matrices

$$C_i = A + t_i B,\ i = 1, 2, \ldots, n + 1,$$

are nilpotent (i.e. $C_i^n = 0$).

Show that both A and B are nilpotent.

Problem 6. (25 points)

Let $p > 1$. Show that there exists a constant $K_p > 0$ such that for every $x, y \in \mathbb{R}$ satisfying $|x|^p + |y|^p = 2$, we have

$$(x - y)^2 \leq K_p(4 - (x + y)^2).$$

1.2.2 Day 2, 1995

Problem 1. (10 points)

Let A be 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u \in \mathbb{R}^3$. Prove that:

a) $A^T = -A$, where A^T denotes the transpose of the matrix A;

b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$, where $v \times u$ denotes the vector product in $\mathbb{R}^3$.

Problem 2. (15 points)

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that $b_0 = 1, b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

Problem 3. (15 points)

Let all roots of an n-th degree polynomial $P(z)$ with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

Problem 4. (15 points)

a) Prove that for every $\epsilon > 0$ there is a positive integer n and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\max_{x\in[-1,1]} \left| x - \sum_{k=1}^{n} \lambda_k x^{2k+1} \right| < \epsilon.$$

b) Prove that for every odd continuous function f on $[-1, 1]$ and for every $\epsilon > 0$ there is a positive integer n and real numbers $\mu_1, \ldots, \mu_n$ such that

$$\max_{x\in[-1,1]} \left| f(x) - \sum_{k=1}^{n} \mu_k x^{2k+1} \right| < \epsilon.$$

Recall that f is odd means that $f(x) = -f(-x)$ for all $x \in [-1, 1]$.

Problem 5. (10+15 points)

a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos x + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$, has positive as well as negative values in the period $[0, 2\pi)$.

b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}} x)$$

has at least 40 zeros in the interval $(0, 1000)$.

Problem 6. (20 points)

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on the interval $[0,1]$ such that

$$\int_0^1 f_m(x)f_n(x)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\sup\{|f_n(x)| : x \in [0,1] \text{ and } n = 1,2,\ldots\} < +\infty.$$

Show that there exists no subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim\limits_{k\to\infty} f_{n_k}(x)$ exists for all $x \in [0,1]$.

1.3 Olympic 1996

1.3.1 Day 1, 1996

Problem 1. (10 points)

Let for $j = 0, ..., n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}$$

Calculate $det(A)$, where $det(A)$ denotes the determinant of A.

Problem 2. (10 points) Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx,$$

where n is a natural number.

Problem 3. (15 points)

The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $dim V = n < \infty$.

(i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(ii) Find the maximal number of distinct pairwise commuting involutions on V.

Problem 4. (15 points)

Let $a_1 = 1,\ a_n = \dfrac{1}{n}\sum\limits_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

(i) $\limsup\limits_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$;

(ii) $\limsup\limits_{n\to\infty} |a_n|^{1/n} \geq \dfrac{2}{3}$.

Problem 5. (25 points)

i) Let a, b be real number such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every x in $[0, 1]$. Prove that

$$\lim_{n\to\infty} n \int_0^1 (1 + ax + bx^2) dx = \begin{cases} -\dfrac{1}{a} & \text{if } a < 0 \\ +\infty & \text{if } a \geq 0. \end{cases}$$

ii) Let $f : [0, 1] \to [0, \infty)$ be a function with a continuous second derivative and let $f''(x) \leq 0$ for every x in [0,1]. Suppose that $L = \lim\limits_{n\to\infty} n \int\limits_0^1 (f(x))^n dx$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

Problem 6. (25 points)

Upper content of a subset E of the plane $\mathbb{R}$ is defined as

$$\mathcal{C}(E) = \inf \left\{ \sum_{i=1}^{n} \operatorname{diam}(E_i) \right\}$$

where inf is taken over all finite of sets $E_1, \ldots, E_n,\ n \in \mathbb{N}$ in $\mathbb{R}^2$ such that $E \subset \bigcup\limits_{i=1}^{n} E_i$. Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{length(L) : L \text{ is a closed line segment onto which } E \text{ can be contracted}\}$$

Show that

(a) $\mathcal{C}(L) = \text{lenght}\,(L)$ if L is a closed line segment;

(b) $\mathcal{C}(E) \geq \mathcal{K}(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2, 2), (2, 2)$ and $(0, 4)$, and T' is its reflexion about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks: All distances used in this problem are Euclidian. Diameter of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x, y) : x, y \in E\}$. Contraction of a set E to a set F is a mapping $f : E \mapsto F$ such that $\operatorname{dist}(f(x), f(y)) \le \operatorname{dist}(x, y)$ for all $x, y \in E$. A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. Triangle is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

1.3.2 Day 2, 1996

Problem 1. (10 points)

Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function, then the sequence of iterates $x_{n+l} = f(x_n)$ converges if and only if

$$\lim_{n \to \infty} (x_{n+1} - x_n) = 0.$$

Problem 2. (10 points)

Let θ be a positive real number and let $cosht = \dfrac{e^t + e^{-t}}{2}$ denote the hyperbolic cosine. Show that if $k \in \mathbb{N}$ and both $coshk\theta$ and $cosh(k+1)\theta$ are rational, then so is $cosh\theta$.

Problem 3. (15 points)

Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let H consist of those matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in G for which $a_{11} = a_{22} = 1$.

(a) Show that H is an abelian subgroup of G.

(b) Show that H is not finitely generated.

Remarks. $GL_2(\mathbb{R})$ denotes, as usual, the group (under matrix multiplication) of all 2×2 invertible matrices with real entries (elements).

Abelian means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

Problem 4. (20 points)

Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any *arc*A of C of length $l(A) \geq \frac{\pi}{2}$.

Problem 5. (20 points)

(i) Prove that

$$\lim_{n\to+\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\Big| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \Big| \leq \frac{c}{x}.$$

Problem 6. (Carleman's inequality) (25 points)

(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n > 0, n = 1, 2, \dots$ and $\sum\limits_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.

(ii) Prove that for every $\epsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n > 0, n = 1, 2, \dots, \sum\limits_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} > (e - \epsilon) \sum_{n=1}^{\infty} a_n.$$

1.4 Olympic 1997

1.4.1 Day 1, 1997

Problem 1.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers, such that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} ln\Big(\frac{k}{n} + \epsilon_n\Big),$$

where ln denotes the natural logarithm.

Problem 2.

Suppose $\sum\limits_{n=1}^{\infty} a_n$ converges. Do the following sums have to converge as well?

a) $a_1 + a_2 + a_4 + a_3 + a_8 + a_7 + a_6 + a_5 + a_{16} + a_{15} + \cdots + a_9 + a_{32} + \cdots$

b) $a_1 + a_2 + a_3 + a_4 + a_5 + a_7 + a_6 + a_8 + a_9 + a_{11} + a_{13} + a_{15} + a_{10} + a_{12} + a_{14} + a_{16} + a_{17} + a_{19} + \cdots$

Justify your answers.

Problem 3.

Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is an invertible matrix then n is divisible by 3.

Problem 4.

Let α be a real number, $1 < \alpha < 2$.

a) Show that α has a unique representation as an infinite product

$$\alpha = \Big(1 + \frac{1}{n_1}\Big)\Big(1 + \frac{1}{n_2}\Big) \ldots$$

b) Show that α is rational if and only if its infinite product has the following property:

For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

Problem 5. For a natural n consider the hyperplane

$$R_0^n = \Big\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\Big\}$$

and the lattice $Z_0^n = \{y \in R_0^n :$ all y_i are integers$\}$. Define the (quasi-)norm in $\mathbb{R}_n$ by $\| x \|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ if $0 < p < \infty$, and $\| x \|_\infty = \max_i |x_i|$.

a) Let $x \in R_0^n$ be such that

$$\max_i x_i - \min_i x_i \leq 1.$$

For every $p \in [1, \infty]$ and for every $y \in Z_0^n$ prove that

$$\| x \|_p \leq \| x + y \|_p .$$

b) For every $p \in (0, 1)$, show that there is an n and an $x \in R_0^n$ with $\max_i x_i - \min_i x_i \leq 1$ and an $y \in Z_0^n$ such that

$$\| x \|_p > \| x + y \|_p .$$

Problem 6. Suppose that F is a family of finite subsets of $\mathbb{N}$ and for any two sets $A, B \in F$ we have $A \cap B \neq \emptyset$.

a) Is it true that there is a finite subset Y of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$?

b) Is the statement a) true if we suppose in addition that all of the members of F have the same size?

Justify your answers.

1.4.2 Day 2, 1997

Problem 1.

Let f be a $C^3(\mathbb{R})$ non-negative function, $f(0) = f'(0) = 0, 0 < f''(0)$. Let

$$g(x) = \left(\frac{\sqrt{f(x)}}{f'(x)} \right)'$$

for $x \neq 0$ and $g(0) = 0$. Show that g is bounded in some neighbourhood of 0. Does the theorem hold for $f \in \mathcal{C}^2(\mathbb{R})$?

Problem 2.

Let M be an invertible matrix of dimension $2n \times 2n$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that $detM.detH = detA$.

Problem 3.

Show that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(logn)}{n^\alpha}$ converges if and only if $\alpha > 0$.

Problem 4.

a) Let the mapping $f : M_n \to \mathbb{R}$ from the space $M_n = \mathbb{R}^{n^2}$ of $n \times n$ matrices with real entries to reals be linear, i.e.:

$$f(A + B) = f(A) + f(B), f(cA) = cf(A) \tag{1}$$

for any $A, B \in M_n, c \in \mathbb{R}$. Prove that there exists a unique matrix $C \in M_n$ such that $f(A) = tr(AC)$ for any $A \in M_n$. (If $A = \{a_{ij}\}_{i,j=1}^n$ then $tr(A) = \sum\limits_{i=1}^{n} a_{ii}$).

b) Suppose in addition to (1) that

$$f(A.B) = f(B.A) \tag{2}$$

for any $A, B \in M_n$. Prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda.tr(A)$.

Problem 5.

Let X be an arbitrary set, let f be an one-to-one function mapping X onto itself. Prove that there exist mappings $g_1, g_2 : X \to X$ such that $f = g_1 \circ g_2$ and $g_1 \circ g_1 = id = g_2 \circ g_2$, where id denotes the identity mapping on X.

Problem 6.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Say that f "crosses the axis" at x if $f(x) = 0$ but in any neighbourhood of x there are y, z with $f(y) < 0$ and $f(z) > 0$.

a) Give an example of a continuous function that "crosses the axis" infiniteley often.

b) Can a continuous function "cross the axis" uncountably often? Justify your answer.

1.5 Olympic 1998

1.5.1 Day 1, 1998

Problem 1. (20 points)

Let V be a 10-dimensional real vector space and U_1 and U_2 two linear subspaces such that $U_1 \subseteq U_2, dim_{\mathbb{R}} U_1 = 3$ and $dim_{\mathbb{R}} U_2 = 6$. Let ϵ be the set of all linear maps $T : V \to V$ which have U_1 and U_2 as invariant subspaces (i.e., $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$). Calculate the dimension of ϵ as a real vector space.

Problem 2. Prove that the following proposition holds for $n = 3$ (5 points) and $n = 5$ (7 points), and does not hold for $n = 4$ (8 points).

"For any permutation π_1 of $\{1, 2, \ldots, n\}$ different from the identity there is a permutation π_2 such that any permutation π can be obtained from π_1 and π_2 using only compositions (for example, $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$)."

Problem 3. Let $f(x) = 2x(1 - x), x \in \mathbb{R}$. Define

$$f(n) = \overbrace{f \circ \cdots \circ f}^{n}.$$

a) (10 points) Find $lim_{n\to\infty} \int_0^1 f_n(x)dx$

b) (10 points) Compute $\int_0^1 f_n(x)dx$ for $n = 1, 2, \ldots$.

Problem 4. (20 points)

The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2, f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0, 1)$ for which

$$f(\xi).f'(\xi) + f''(\xi) = 0.$$

Problem 5. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.

a) (15 points) Prove that for every real x the following inequality holds:

$$(n-1)(P'(x))^2 \geq nP(x)P''(x). \tag{2}$$

b) (5 points) Examine the cases of equality.

Problem 6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y) + yf(x) \leq 1.$$

a) (15 points) Show that

$$\int_0^1 f(x)dx \leq \frac{\pi}{4}.$$

b) (5 points) Find a function, satisfying the condition, for which there is equality.

1.5.2 Day 2, 1998

Problem 1. (20 points)

Let V be a real vector space, and let $f, f_1, \dots, f_k$ be linear maps from V to $\mathbb{R}$ Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \cdots = f_k(x) = 0$. Prove that f is a linear combination of $f_1, f_2, \dots, f_k$.

Problem 2. (20 points) Let

$$\mathcal{P} = \{f : f(x) = \sum_{k=0}^{3} a_k x^k,\ a_k \in \mathbb{R}, |f(\pm 1)| \leq 1, |f(\pm \frac{1}{2})| \leq 1\}$$

Evaluate

$$\sup_{f\in\mathcal{P}} \max_{-1\leq x\leq 1} |f''(x)|$$

and find all polynomials $f \in \mathcal{P}$ for which the above "sup" is attained.

Problem 3. (20 points) Let $0 < c < 1$ and

$$f(x) = \begin{cases} \dfrac{x}{c} & \text{for } x \in [0,c], \\ \dfrac{1-x}{1-c} & \text{for } x \in [c,1]. \end{cases}$$

We say that p is an n-periodic point if

$$\underbrace{f(f(\dots f(}_{n}p))) = p$$

and n is the smallest number with this property. Prove that for every $n \geq 1$ the set of n-periodic points is non-empty and finite.

Problem 4. (20 points) Let $A_n = \{1, 2, \dots, n\}$, where $n \geq 3$. Let $\mathcal{F}$ be the family of all non-constant functions $f : A_n \to A_n$ satisfying the following conditions:

(1) $f(k) \leq f(k+1)$ for $k = 1, 2, \dots, n-1$,

(2) $f(k) = f(f(k+1))$ for $k = 1, 2, \dots, n-1$. Find the number of functions in $\mathcal{F}$.

Problem 5. (20 points)

Suppose that $\mathcal{S}$ is a family of spheres (i.e., surfaces of balls of positive radius) in $\mathbb{R}^2, n \geq 2$, such that the intersection of any two contains at most one point. Prove that the set M of those points that belong to at least two different spheres from $\mathcal{S}$ is countable.

Problem 6. (20 points) Let $f : (0,1) \to [0, \infty)$ be a function that is zero except at the distinct points $a_1, a_2, \dots$. Let $b_n = f(a_n)$.

(a) Prove that if $\sum\limits_{n=1}^{\infty} b_n < \infty$, then f is differentiable at at least one point $x \in (0,1)$.

(b) Prove that for any sequence of non-negative real numbers $(b_n)_{n=1}^{\infty}$ with $\sum\limits_{n=1}^{\infty} b_n = \infty$, there exists a sequence $(a_n)_{n=1}^{\infty}$ such that the function f defined as above is nowhere differentiable.

1.6 Olympic 1999

1.6.1 Day 1, 1999

Problem 1.

a) Show that for any $m \in \mathbf{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix. (6 points)

b) Show that $detA > 0$ for every real $m \times m$ matrix satisfying $A^3 = A + I$. (14 points)

Problem 2. Does there exist a bijective map $\pi : \mathbf{N} \to \mathbf{N}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

(20 points)

Problem 3. Suppose that a function $f : \mathbf{R} \to \mathbf{R}$ satisfies the inequality

$$\Big| \sum_{k=1}^{n} 3^k (f(x+ky) - f(x-ky)) \Big| \leq 1 \tag{1}$$

for every positive integer n and for all $x, y \in \mathbf{R}$. Prove that f is a constant function. (20 points)

Problem 4. Find all strictly monotonic functions $f : (0, +\infty) \to (0, +\infty)$ such that $f\Big(\dfrac{x^2}{f(x)}\Big) \equiv x$. (20 points)

Problem 5.

Suppose that $2n$ points of an $n \times n$ grid are marked. Show that for some $k > l$ one can select $2k$ distinct marked points, say $a_1, \ldots, a_{2k}$, such that a_1 and a_2 are in the same row, a_2 and a_3 are in the same column, ... , a_{2k-l} and a_{2k} are in the same row, and a_{2k} and a_1 are in the same column. (20 points)

Problem 6.

a) For each $1 < p < \infty$ find a constant $c_p < \infty$ for which the following statement holds: If $f : [-1, 1] \to \mathbf{R}$ is a continuously differentiable function satisfying $f(1) > f(-1)$ and $|f'(y)| \leq 1$ for all $y \in [-1, 1]$, then there is an $x \in [-1, 1]$ such that $f'(x) > 0$ and $|f(y) - f(x)| \leq c_p (f'(x))^{1/p} |y - x|$ for all $y \in [-1, 1]$. (10 points)

b) Does such a constant also exist for $p = 1$? (10 points)

1.6.2 Day 2, 1999

Problem 1. Suppose that in a not necessarily commutative ring R the square of any element is 0. Prove that $abc + abc = 0$ for any three elements a, b, c. (20 points)

Problem 2. We throw a dice (which selects one of the numbers $1, 2, \ldots, 6$ with equal probability) n times. What is the probability that the sum of the values is divisible by 5? (20 points)

Problem 3.

Assume that $x_1, \ldots, x_n \geq -1$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove that $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$. (20 points)

Problem 4. Prove that there exists no function $f : (0, +\infty) \to (0, +\infty)$ such that $f^2(x) \geq f(x+y)(f(x)+y)$ for any $x, y > 0$. (20 points)

Problem 5. Let S be the set of all words consisting of the letters x, y, z, and consider an equivalence relation $\sim$ on S satisfying the following conditions: for arbitrary words $u, v, w \in S$

(i) $uu \sim u$;

(ii) if $v \sim w$, then $uv \sim uw$ and $vu \sim wu$.

Show that every word in S is equivalent to a word of length at most 8. (20 points)

Problem 6. Let A be a subset of $\mathbf{Z}_n = \frac{\mathbf{Z}}{n\mathbf{Z}}$ containing at most $\frac{1}{100} \ln n$ elements. Define the rth Fourier coefficient of A for $r \in \mathbf{Z}_n$ by

$$f(r) = \sum_{s \in A} exp\Big(\frac{2\pi i}{n} sr\Big).$$

Prove that there exists an $r \neq 0$, such that $|f(r)| \geq \frac{|A|}{2}$. (20 points)

1.7 Olympic 2000

1.7.1 Day 1, 2000

Problem 1.

Is it true that if $f : [0,1] \to [0,1]$ is

a) monotone increasing

b) monotone decreasing then there exists an $x \in [0,1]$ for which $f(x) = x$?

Problem 2.

Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs (w, z) of complex numbers with $w \neq z$ for which $p(w) = p(z)$ and $q(w) = q(z)$.

Problem 3.

A and B are square complex matrices of the same size and

$$rank(AB - BA) = 1.$$

Show that $(AB - BA)^2 = 0$.

Problem 4.

a) Show that if (x_i) is a decreasing sequence of positive numbers then

$$\Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant C so that if (x_i) is a decreasing sequence of positive numbers then

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \Big(\sum_{i=m}^{\infty} x_i^2\Big)^{1/2} \leq C \sum_{i=1}^{\infty} x_i.$$

Problem 5.

Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e + f + g = 0$. Show that $e = f = g = 0$.

(R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a = 0$. An idempotent x is an element satisfying $x = x^2$.)

Problem 6.

Let $f : \mathbb{R} \to (0, \infty$ be an increasing differentiable function for which $\lim\limits_{x\to\infty} f(x) = \infty$ and f' is bounded.

Let $F(x) = \int\limits_0^x f$. Define the sequence (a_n) inductively by

$$a_0 = 1,\ a_{n+1} = a_n + \frac{1}{f(a_n)},$$

and the sequence (b_n) simply by $b_n = F^{-1}(n)$. Prove that $\lim_{n\to\infty}(a_n - b_n) = 0$.

1.7.2 Day 2, 2000

Problem 1.

a) Show that the unit square can be partitioned into n smaller squares if n is large enough.

b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

Problem 2. Let f be continuous and nowhere monotone on $[0, 1]$. Show that the set of points on which f attains local minima is dense in $[0, 1]$.

(A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

Problem 3. Let $p(z)$ be a polynomial of degree n with complex coefficients. Prove that there exist at least $n+1$ complex numbers z for which $p(z)$ is 0 or 1.

Problem 4. Suppose the graph of a polynomial of degree 6 is tangent to a straight line at 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.

a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are equal as well.

b) Let $k = \dfrac{A_2A_3}{A_1A_2}$ and let K be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}k^5 < K < \frac{7}{2}k^5.$$

Problem 5. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$

$$f(x)f(yf(x)) = f(x + y).$$

Problem 6. For an $m \times m$ real matrix A, e^A is defined as $\sum\limits_{n=0}^{\infty} \frac{1}{n!} A^n$. (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and $B, p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)

1.8 Olympic 2001

1.8.1 Day 1, 2001

Problem 1.

Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left-to-right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Problem 2.

Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary noncommutative group?

Problem 3. Find $\lim\limits_{t \nearrow 1}(1-t) \sum\limits_{n=1}^{\infty} \frac{t^n}{1+t^n}$, where $t \nearrow 1$ means that t approaches 1 from below.

Problem 4.

Let k be a positive integer. Let $p(x)$ be a polynomial of degree n each of whose coefficients is $-1, 1$ or 0, and which is divisible by $(x-1)^k$. Let q be a prime such that $\frac{q}{\ln q} < \frac{k}{ln(n+1)}$. Prove that the complex qth roots of unity are roots of the polynomial $p(x)$.

Problem 5.

Let A be an $n \times n$ complex matrix such that $A \neq \lambda I$ for all $\lambda \in \mathbf{C}$.

Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Problem 6.

Suppose that the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0, f'(x) \geq 0, g(x) > 0, g'(x) > 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x\to\infty} a(x) = A > 0, \ \lim_{x\to\infty} b(x) = B > 0, \ \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}.$$

1.8.2 Day 2, 2001

Problem 1.

Let $r, s \geq 1$ be integers and $a_0, a_1 \dots, a_{r-1}, b_0, b_1, \dots, b_{s-1}$ be real non-negative numbers such that

$$(a_0+a_1x+a_2x^2+\dots+a_{r-1}x^{r-1}+x^r)(b_0+b_1x+b_2x^2+\dots+b_{s-1}x^{s-1}+x^s)$$
$$= 1 + x + x^2 + \dots + x^{r+s-1} + x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Problem 2.

Let $a_0 = \sqrt{2}, \ b_0 = 2, \ a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, \ b_{n+1} = \dfrac{2b_n}{2 + \sqrt{4 + b_n^2}}.$

a) Prove that the sequences $(a_n), (b_n)$ are decreasing and converge to 0.

b) Prove that the sequence $(2^n a_n)$ is increasing, the sequence $(2^n b_n)$ is decreasing and that these two sequences converge to the same limit.

c) Prove that there is a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \dfrac{C}{8^n}$.

Problem 3.

Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

Problem 4.

Let $A = (a_{k,l})_{k,l=1,\dots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \dots, n\}$ and $1 \le j_1 < \cdots < j_m \le n$ the determinant of the matrix $(a_{jk,jl})_{k,l=1,\dots,m}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix

$$(a_{\sigma(k),\sigma(l)})_{k,l=1\dots,n}$$

has all of its nonzero elements above the diagonal.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x + y) \ge f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}.$$

Problem 6.

For each positive integer n, let $f_n(\vartheta) = \sin\vartheta. \sin(2\vartheta). \sin(4\vartheta) \dots \sin(2^n\vartheta)$. For all real ϑ and all n, prove that

$$|f_n(\vartheta)| \le \frac{2}{\sqrt{3}}\left|f_n\left(\frac{\pi}{\sqrt{3}}\right)\right|.$$

1.9 Olympic 2002

1.9.1 Day 1, 2002

Problem 1. A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x axis.

Prove that standard parabolas with vertices s $(A_1), s(A_2), s(A_3)$ intersect pairwise at the points $s(V_1), s(V_2), s(V_3)$.

Problem 2. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Problem 3. Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \; b_k = 2^{k-n}, \; \text{for } k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} = 0. \tag{1}$$

Problem 4. Let $f : [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, \ldots$. Suppose that the set $T_p = \{p_n : n = 0, 1, 2, \ldots\}$ is closed, i.e., if $x \notin T_p$ then there is a $\delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$. Show that T_p has finitely many elements.

Problem 5. Prove or disprove the following statements:

(a) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.

(b) There exists a continuously differentiable function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.

Problem 6. For an $n \times n$ matrix M with real entries let $\| M \| = \sup\limits_{x \in \mathbb{R}^n \setminus \{0\}} \dfrac{\| Mx \|_2}{\| x \|_2}$, where $\| \, . \, \|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrix A with real entries satisfies $\| A^k - A^{k-1} \| \leq \dfrac{1}{2002k}$ for all positive integers k. Prove that $\| A^k \| \leq 2002$ for all positive integers k.

1.9.2 Day 2, 2002

Problem 1. Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if } i \neq j \\ 2, & \text{if } i = j. \end{cases}$$

Problem 2. Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Problem 3. For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \; b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n.b_n$ is an integer.

Problem 4. In the tetrahedron $OABC$, let $\angle BOC = \alpha, \angle COA = \beta$ and $\angle AOB = \gamma$. Let σ be the angle between the faces OAB and OAC, and let τ be the angle between the faces OBA and OBC. Prove that

$$\gamma > \beta. \cos \sigma + \alpha \cos \tau.$$

Problem 5. Let A be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \;\Leftrightarrow\; \exists S \in GL_n(\mathbb{C} \text{ such that } A = S\overline{S}^{-1}.$$

(If $A = [a_{ij}]$ then $\overline{A} = [\overline{a_{ij}}]$, where $\overline{a_{ij}}$ is the complex conjugate of $a_{ij}; GL_n(\mathbb{C})$ denotes the set of all $n \times n$ invertible matrices with complex entries, and I_n is the identity matrix.)

Problem 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists L > 0 \; \forall x_1, x_2 \in \mathbb{R}^n \;\; \| \nabla f(x_1) - \nabla f(x_2) \| \leq L \| x_1 - x_2 \| .$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n \;\; \| \nabla f(x_1) - \nabla f(x_2) \|^2 \leq L < \nabla f(x_1) - \nabla f(x_2), x_1 - x_2 > . \tag{1}$$

In this formula $< a, b >$ denotes the scalar product of the vectors a and b.

1.10 Olympic 2003

1.10.1 Day 1, 2003

Problem 1.

a) Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \dfrac{3}{2} a_n$ for all n. Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity. (10 points)

b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \ldots$ with the same properties such that

$$\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha.$$

(10 points)

Problem 2. Let $a_1, a_2, \ldots, a_{51}$ be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, \ldots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that $\underbrace{x + x + \cdots + x}_{p} = 0$ for any element x of the field. If there exists no such p, the characteristic is 0.) (20 points)

Problem 3. Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.) (20 points)

Problem 4. Determine the set of all pairs (a, b) of positive integers for which the set of positive integers can be decomposed into two sets A and B such that $a.A = b.B$. (20 points)

Problem 5. Let $g : [0,1] \to \mathbb{R}$ be a continuous function and let $f_n : [0,1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x} \int\limits_0^x f_n(t)dt \ (x \in (0,1],\ n = 0, 1, 2, \ldots).$$

Determine $\lim\limits_{n\to\infty} f_n(x)$ for every $x \in (0,1]$. (20 points)

Problem 6. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with real coefficients. Prove that if all roots of f lie in the left half-plane $\{z \in \mathbb{C} : Rez < 0\}$ then

$$a_k a_{k+3} < a_{k+1} a_{k+2}$$

holds for every $k = 0, 1, ..., n-3$. (20 points)

1.10.2 Day 2, 2003

Problem 1. Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

2. Evaluate the limit

$$\lim_{x\to 0+} \int_x^{2x} \frac{\sin^m t}{t^n} dt \quad (m, n \in \mathbb{N}).$$

Problem 3. Let A be a closed subset of $\mathbb{R}^n$ and let B be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that

$$|a_0 - b| = \inf_{a \in A} |a - b|.$$

Prove that B is dense in $\mathbb{R}^n$; that is, the closure of B is $\mathbb{R}^n$.

Problem 4. Find all positive integers n for which there exists a family $\mathcal{F}$ of three-element subsets of $S = \{1, 2, \ldots, n\}$ satisfying the following two conditions:

(i) for any two different elements $a, b \in S$, there exists exactly one $A \in \mathcal{F}$ containing both a, b;

(ii) if a, b, c, x, y, z are elements of S such that if $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, then $\{x, y, z\} \in \mathcal{F}$.

Problem 5. a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ there exists a function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{Q}$.

b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

Problem 6. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$a_0 = 1,\ a_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n}\frac{a_k}{n-k+2}.$$

Find the limit

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{a_k}{2^k},$$

if it exists.

1.11 Olympic 2004

1.11.1 Day 1, 2004

Problem 1. Let S be an infinite set of real numbers such that $|s_1 + s_2 + \cdots + s_k| < 1$ for every finite subset $\{s_1, s_2, \ldots, s_k\} \subset S$. Show that S is countable. [20 points]

Problem 2. Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\ldots(P}_{2004}(x))\ldots)) = 0?$$

[20 points]

Problem 3. Let S_n be the set of all sums $\sum\limits_{k=1}^{n} x_k$, where $n \geq 2,\ 0 \leq x_1, x_2, \ldots, x_n \leq \dfrac{\pi}{2}$ and

$$\sum_{k=1}^{n}\sin x_k = 1.$$

a) Show that S_n is an interval. [10 points]

b) Let l_n be the length of S_n. Find $\lim\limits_{n\to\infty} l_n$. [10 points]

Problem 4. Suppose $n \geq 4$ and let M be a finite set of n points in $\mathbb{R}^3$, no four of which lie in a plane. Assume that the points can be coloured black or white so that any sphere which intersects M in at least four points has the property that exactly half of the points in the intersection of M and the sphere are white. Prove that all of the points in M lie on one sphere. [20 points]

Problem 5. Let X be a set of $\binom{2k-4}{k-2}+1$ real numbers, $k \le 2$. Prove that there exists a monotone sequence $\{x_n\}_{i=1}^{k} \supseteq X$ such that

$$|x_{i+1} - x_1| \ge 2|x_i - x_1|$$

for all $i = 2, \dots, k-1$. [20 points]

Problem 6. For every complex number $z \ne \{0, 1\}$ define

$$f(z) := \sum (\log z)^{-4},$$

where the sum is over all branches of the complex logarithm.

a) Show that there are two polynomials P and Q such that $f(z) = \dfrac{P(z)}{Q(z)}$ for all $z \in \mathbb{C} \backslash \{0, 1\}$. [10 points]

b) Show that for all $z \in \mathbb{C} \backslash \{0, 1\}$

$$f(z) = z\frac{z^2 + 4z + 1}{6(z-1)^4}.$$

[10 points]

1.11.2 Day 2, 2004

Problem 1. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find BA. [20 points]

Problem 2. Let $f, g : [a, b] \to [0, \infty)$ be continuous and non-decreasing functions such that for each $x \in [a, b]$ we have

$$\int\limits_a^x \sqrt{f(t)}dt \le \int\limits_a^x \sqrt{g(t)}dt$$

and $\int\limits_a^b \sqrt{f(t)}dt = \int\limits_a^b \sqrt{g(g)}dt$.

Prove that $\int\limits_a^b \sqrt{1+f(t)}dt \geq \int\limits_a^b \sqrt{1+g(t)}dt$. [20 points]

Problem 3. Let D be the closed unit disk in the plane, and let $p_1, p_2, \ldots, p_n$ be fixed points in D. Show that there exists a point p in D such that the sum of the distances of p to each of $p_1, p_2, \ldots, p_n$ is greater than or equal to 1. [20 points]

Problem 4. For $n \geq 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $m_1, m_2, \ldots, m_k$, respectively. Consider the linear operator L_M defined by $L_M(X) = MX + XM^T$, for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose of M; that is, if $M = (m_{k,l})$, then $M^T = (m_{l,k})$.) [20 points]

Problem 5. Prove that

$$\int\limits_0^1 \int\limits_0^1 \frac{dxdy}{x^{-1} + |\ln y| - 1} \leq 1.$$

[20 points]

Problem 6. For $n \geq 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that $S(A_k^{n-1}) = S(A_k^{n-1})$ for every $n, k \geq 1$. [20 points]

1.12 Olympic 2005

1.12.1 Day 1, 2005

Problem 1. Let A be the $n \times n$ matrix, whose $(i,j)^{th}$ entry is $i+j$ for all $i, j = 1, 2, \ldots, n$. What is the rank of A?

Problem 2. For an integer $n \geq 3$ consider the sets

$$S_n = \{(x_1, x_2, \ldots, x_n) : \forall i \; x_i \in \{0, 1, 2\}\}$$
$$A_n = \{(x_1, x_2, \ldots, x_n) \in S_n : \; \forall i \leq n-2 \; |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\}$$

and

$$B_n = \{(x_1, x_2, \ldots, x_n) \in S_n : \; \forall i \le n-1 \; (x_i = x_{i+1} \Rightarrow x_i \neq 0)\}.$$

Prove that $|A_{n+1} = 3.|B_n|$. ($|A|$ denotes the number of elements of the set A.)

Problem 3. Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that

$$\Big| \int_0^1 f^3(x)dx - f^2(0) \int_0^1 f(x)dx \Big| \le \max_{0 \le x \le 1} |f'(x)| \Big(\int_0^1 f(x)dx \Big)^2.$$

Problem 4. Find all polynomials $P(x) = a_n x^n + a_{n-1} x_{n-1} + \cdots + a_1 x + a_0$ ($a_n \neq 0$) satisfying the following two conditions:

(i) $(a_0, a_1, \ldots, a_n)$ is a permutation of the numbers $(0, 1, ..., n)$ and

(ii) all roots of $P(x)$ are rational numbers.

Problem 5. Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \le 1$$

for all x. Prove that $\lim_{x \to \infty} f(x) = 0$.

Problem 6. Given a group G, denote by $G(m)$ the subgroup generated by the m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that $G(gcd(m, n))$ is also commutative. ($gcd(m, n)$ denotes the greatest common divisor of m and n.)

1.12.2 Day 2, 2005

Problem 1. Let $f(x) = x^2 + bx + c$, where b and c are real numbers, and let

$$M = \{x \in \mathbb{R} : \; |f(x)| < 1\}.$$

Clearly the set M is either empty or consists of disjoint open intervals. Denote the sum of their lengths by $|M|$. Prove that

$$|M| \le 2\sqrt{2}.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every $n = 2, 3, \ldots$. Does it follow that f is a polynomial?

Problem 3. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subs pace V such that

$$\forall X, Y \in V \ \mathrm{trace}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

Problem 4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

Problem 5. Find all $r > 0$ such that whenever $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $|\mathrm{grad}\, f(0,0)| = 1$ and $|\mathrm{grad}\, f(u) - \mathrm{grad}\, f(v)| \leq |u - v|$ for all $u, v \in \mathbb{R}^2$, then the maximum of f on the disk $\{u \in \mathbb{R}^2 : |u| \leq r\}$ is attained at exactly one point. ($\mathrm{grad}\, f(u) = (\partial_1 f(u), \partial_2 f(u))$ is the gradient vector of f at the point u. For a vector $u = (a, b), |u| = \sqrt{a^2 + b^2}$.)

Problem 6. Prove that if p and q are rational numbers and $r = p + q\sqrt{7}$, then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with integer entries and with $ad - bc = 1$ such that

$$\frac{ar + b}{cr + d} = r.$$

1.13 Olympic 2006

1.13.1 Day 1, 2006

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.

a) If f is continuous and $range(f) = \mathbb{R}$ then f is monotonic.

b) If f is monotonic and $range(f) = \mathbb{R}$ then f is continuous.

c) If f is monotonic and f is continuous then $range(f) = \mathbb{R}$.

(20 points)

Problem 2. Find the number of positive integers x satisfying the following two conditions:

1. $x < 10^{2006}$;

2. $x^2 - x$ is divisible by 10^{2006}.

(20 points)

Problem 3. Let A be an $n \times n$-matrix with integer entries and $b_1, \ldots, b_k$ be integers satisfying $detA = b_1 \ldots b_k$. Prove that there exist $n \times n$-matrices $B_1, \ldots, B_k$ with integer entries such that $A = B_1 \ldots B_k$ and $detB_i = b_i$ for all $i = 1, \ldots, k$. (20 points)

Problem 4. Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial. (20 points)

Problem 5. Let $a, b, c, d, e > 0$ be real numbers such that $a^2 + b^2 + c^2 = d^2 + e^2$ and $a^4 + b^4 + c^4 = d^4 + e^4$. Compare the numbers $a^3 + b^3 + c^3$ and $d^3 + e^3$. (20 points)

Problem 6. Find all sequences $a_0, a_1, \ldots, a_n$ of real numbers where $n \geq 1$ and $a_n \neq 0$, for which the following statement is true:

If $f : \mathbb{R} \to \mathbb{R}$ is an n times differentiable function and $x_0 < x_1 < \cdots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$ then there exists an $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \cdots + a_n f^{(n)}(h) = 0.$$

(20 points)

1.13.2 Day 2, 2006

Problem 1. Let V be a convex polygon with n vertices.

a) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.

b) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.
(20 points)

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.
(20 points)

Problem 3. Compare $\tan(\sin x)$ and $\sin(\tan x)$ for all $x \in (0, \frac{\pi}{2})$.
(20 points)

Problem 4. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \ldots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \le i, j \le n+1$. Prove that $v_1, \ldots, v_{n+1}$ are linearly dependent over the rationals.
(20 points)

Problem 5. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$(x + m)^3 = nx$$

has three distinct integer roots.
(20 points)

Problem 6. Let $A_i, B_i, S_i (i = 1, 2, 3)$ be invertible real 2×2 matrices such that

1) not all A_i have a common real eigenvector;
2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;
3) $A_1 A_2 A_3 = B_1 B_2 B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.
(20 points)

1.14 Olympic 2007

1.14.1 Day 1, 2007

Problem 1. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

Problem 2. Let $n \geqslant 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

Problem 3. Call a polynomial $P(x_1, \ldots, x_k)$ good if there exist 2×2 real matrices $A_1, \ldots, A_k$ such that

$$P(x_1, \ldots, x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.

(A polynomial is homogeneous if each term has the same total degree.)

Problem 4. Let G be a finite group. For arbitrary sets $U, V, W \subset G$, denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ for which xyz is the unity.

Suppose that G is partitioned into three sets A, B and C (i.e. sets A, B, C are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

Problem 5. Let n be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum\limits_{i=1}^{n} f(k + a_i l) = 0$ whenever k and l are integers and $l \neq 0$. Prove that $f = 0$.

Problem 6. How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z) \leqslant 2|$ for any complex number z of unit length?

1.14.2 Day 2, 2007

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

Problem 2. Let x, y and z be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that S divisible by 29^4.

Problem 3. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in \mathcal{C}$ such that $f(p) = p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \leqslant g(y)$ for all $x \leqslant y$.)

Problem 4. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1,\ldots,n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

Find $\det A$.

Problem 6. Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \ldots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geqslant 0$. Prove that there exists a number N such that for every $n \geqslant N$, all roots of f_n are real.

1.15 Olympic 2008

1.15.1 Day 1, 2008

Problem 1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y such that $x - y$ is rational.

Problem 2. Denote by V the real vector space of all real polynomials in one variable, and let $P : V \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$ we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = cf(x_0)$ for all $f \in V$.

Problem 3. Let p be a polynomial with integer coefficients and let $a_1 < a_2 < \cdots < a_k$ be integers.

a) Prove that there exists $a \in \mathbb{Z}$ such that $p(a_i)$ divides $p(a)$ for all $i = 1, 2, \ldots, k$.

b) Does there exist an $a \in \mathbb{Z}$ such that the product $p(a_1).p(a_2)\ldots p(a_k)$ divides $p(a)$?

Problem 4. We say a triple (a_1, a_2, a_3) of nonnegative reals is better than another triple (b_1, b_2, b_3) if two out of the three following inequalities $a_1 > b_1, a_2 > b_2, a_3 > b_3$ are satisfied. We call a triple (x, y, z) special if x, y, z are nonnegative and $x+y+z = 1$. Find all natural numbers n for which there is a set S of n special triples such that for any given special triple we can find at least one better triple in S.

Problem 5. Does there exist a finite group G with a normal subgroup H such that $|\text{Aut } H| > |\text{Aut } G|$?

Problem 6. For a permutation $\sigma = (i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ define $D(\sigma) = \sum\limits_{k=1}^{n} |i_k - k|$. Let $Q(n, d)$ be the number of permutations σ of $(1, 2, \ldots, n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geqslant 2n$.

1.15.2 Day 2, 2008

Problem 1. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

Problem 2. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

Problem 3. Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0 \leqslant k < \frac{n}{2}} \binom{n}{2k+1} 5^k.$$

Problem 4. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x), g(x) \in \mathbb{Z}[x]$ be nonconstant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

Problem 5. Let n be a posotive integer, and consider the matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

Problem 6. Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.

Chapter 2

Solutions

2.1 Solutions of Olympic 1994

2.1.1 Day 1

Problem 1.

Denote by a_{ij} and b_{ij} the elements of A and A^{-1}, respectively. Then for $k \neq m$ we have $\sum\limits_{i=0}^{n} a_{ki}b_{im} = 0$ and from the positivity of a_{ij} we conclude that at least one of $\{b_{im} : 1, 2, \dots, n\}$ is positive and at least one is negative. Hence we have at least two non-zero elements in every column of A^{-1}. This proves part a). For part b) all b_{ij} are zero except $b_{1,1} = 2, b_{n,n} = (-1)^n, b_{i,i+1} = b_{i+1,i} = (-1)^i$ for $i = 1, 2, \dots, n-1$.

Problem 2. From the inequality we get

$$\frac{d}{dx}(\tan^{-1} f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$

for $x \in (a, b)$. Thus $\tan^{-1} f(x) + x$ is non-decreasing in the interval and using the limits we get $\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$. Hence $b - a \geq pi$. One has equality for $f(x) = cotgx, a = 0, b = \pi$.

Problem 3. Let $\mathbb{I}$ be the set of irrational numbers, $\mathbb{Q}$-the set of rational numbers, $\mathbb{Q}^+ = \mathbb{Q} \cup [0, \infty)$. We work by induction. For $n = 1$ the statement is trivial. Let it be true for $n - 1$. We start to prove it for n. From the induction argument there are $n - 1$ different elements $x_1, x_2, \dots, x_{n-1} \in S$ such that

$$\begin{gathered} a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} \in \mathbb{I} \\ forall\ a_1, a_2, \dots, a_n \in \mathbb{Q}^+\ with\ a_1 + a_2 + \cdots + a_{n-1} > 0 \end{gathered} \tag{1}$$

Denote the other elements of S by $x_n, x_{n+1}, \dots, x_{2n-1}$. Assume the statement is not true for n. Then for $k = 0, 1, \dots, n-1$ there are $r_k \in \mathbb{Q}$ such that

$$\sum_{i=1}^{n-1} b_{ik} x_i + c_k x_{n+k} = r_k \text{ for some } b_{ik}, c_k \in \mathbb{Q}^+, \sum_{i=1}^{n-1} b_{ik} + c_k > 0. \tag{2}$$

Also

$$\sum_{k=0}^{n-1} d_k x_{n+k} = R \text{ for some } d_k \in \mathbb{Q}^+,\ \sum_{k=0}^{n-1} d_k > 0,\ R \in \mathbb{Q}. \tag{3}$$

If in (2) $c_k = 0$ then (2)contradicts (1). Thus $c_k \neq 0$ and without loss of generality one may take $c_k = 1$. In (2) also $\sum\limits_{i=1}^{n-1} b_{ik} > 0$ in view of $x_{n+k} \in \mathbb{I}$. Replacing (2) in (3) we get

$$\sum_{k=0}^{n-1} d_k \Big(-\sum_{i=1}^{n-1} b_{ik} x_i + r_k \Big) = R \ \ \textit{or} \ \ \sum_{i=1}^{n-1} \Big(\sum_{k=0}^{n-1} d_k b_{ik} \Big) x_i \in \mathbb{Q},$$

which contradicts (1) because of the condition on $b's$ and $d's$.

Problem 4. For a) using the assumptions we have

$$\begin{aligned} F^k \circ G - G \circ F^k &= \sum_{i=1}^{k} (F^{k-i+1} \circ G \circ F^{i-1} - F^{k-i} \circ G \circ F^i) = \\ &= \sum_{i=1}^{k} F^{k-i} \circ (F \circ G - G \circ F) \circ F^{i-1} = \\ &= \sum_{i=1}^{k} F^{k-i} \circ \alpha F \circ F^{i-1} = \alpha k F^k. \end{aligned}$$

b) Consider the linear operator $L(F) = F \circ G - G \circ F$ acting over all $n \times n$ matrices F. It may have at most n^2 different eigenvalues. Assuming that $F^k \neq 0$ for every k we get that L has infinitely many different eigenvalues αk in view of a) -a contradiction.

Problem 5. Set $\| g \|_1 = \int\limits_0^b |g(x)| dx$ and

$$\omega(f, t) = \sup\{|f(x) - f(y)| : x, y \in [0, b], |x - y| \le t\}.$$

In view of the uniform continuity of f we have $\omega(f,t) \to 0$ as $t \to 0$. Using the periodicity of g we get

$$\int_0^b f(x)g(nx)dx = \sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} f(x)g(nx)dx$$

$$= \sum_{k=1}^n f(bk/n) \int_{b(k-1)/n}^{bk/n} g(nx)dx + \sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} \{f(x) - f(bk/n)\}g(nx)dx$$

$$= \frac{1}{n}\sum_{k=1}^n f(bk/n) \int_0^b g(x)dx + O(\omega(f,b/n) \parallel g \parallel_1)$$

$$= \frac{1}{n}\sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} f(x)dx \int_0^b g(x)dx$$

$$+ \frac{1}{b}\sum_{k=1}^n \left(\frac{b}{n}f(bk/n) - \int_{b(k-1)/n}^{bk/n} f(x)dx\right) \int_0^b g(x)dx + O(\omega(f,b/n) \parallel g \parallel_1)$$

$$= \frac{1}{b}\int_0^b f(x)dx \int_0^b g(x)dx + O(\omega(f,b/n) \parallel g \parallel_1).$$

This proves a). For b) we set $b = \pi$, $f(x) = \sin x$, $g(x) = (1+3\cos^2 x)^{-1}$. From a) and

$$\int_0^\pi \sin x dx = 2, \quad \int_0^\pi (1 + 3\cos^2 x)^{-1} dx = \frac{\pi}{2}$$

we get

$$\lim_{n\to\infty} \int_0^\pi \frac{\sin x}{1 + 3\cos^2 nx} dx = 1.$$

Problem 6. a) For $i = 1, 2, \ldots, k$ we have

$$b_i = f(m_i) - f(m_{i-1}) = (m_i - m_{i-1})f'(x_i)$$

for some $x_i \in (m_{i-1}, m_i)$. Hence $\dfrac{b_i}{a_i} = f'(x_i)$ and so $-1 < \dfrac{b_i}{a_i} < 1$. From the convexity of f we have that f' is increasing and $\dfrac{b_i}{a_i} = f'(x_i) < f'(x_{i+1}) = \dfrac{b_{i+1}}{a_{i+1}}$ because of $x_i < m_i < x_{i+1}$.

b) Set $S_A = \{j \in \{0, 1, \ldots, k\} : a_j > A\}$. Then

$$N \geq m_k - m_0 = \sum_{i=1}^{k} a_i \geq \sum_{j \in S_A} a_j > A|S_A|$$

and hence $|S_A < \dfrac{N}{A}$.

c) All different fractions in $(-1, 1)$ with denominators less or equal A are no more $2A^2$. Using b) we get $k < \dfrac{N}{A} + 2A^2$. Put $A = N^{1/3}$ in the above estimate and get $k < 3N^{2/3}$.

2.1.2 Day 2

Problem 1. Assume that there is $y \in (a, b]$ such that $f(y) \neq 0$. Without loss of generality we have $f(y) > 0$. In view of the continuity of f there exists $c \in [a, y)$ such that $f(c) = 0$ and $f(x) > 0$ for $x \in (c, y]$. For $x \in (c, y]$ we have $|f'(x)| \leq \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c, y]$ because of $g'(x) = \dfrac{f'(x)}{f(x)} - \lambda \leq 0$. Thus $\ln f(x) - \lambda x \geq \ln f(y) - \lambda y$ and $f(x) \geq e^{\lambda x - \lambda y} f(y)$ for $x \in (c, y]$. Thus

$$0 = f(c) = f(c + 0) \geq e^{\lambda c - \lambda y} f(y) > 0$$

Problem 2. We have $f(1, 0) = e^{-1}, f(0, 1) = -e^{-1}$ and $te^{-t} \leq 2e^{-2}$ for $t \geq 2$. Therefore $|f(x, y)| \leq (x^2 + y^2)e^{-x^2-y^2} \leq 2e^{-2} < e^{-1}$ for $(x, y) \neq M = \{(u, v) : u^2 + v^2 \leq 2\}$ and f cannot attain its minimum and its maximum outside M. Part a) follows from the compactness of M and the continuity of f. Let (x, y) be a point from part b). From $\dfrac{\partial f}{\partial x}(x, y) = 2x(1 - x^2 + y^2)e^{-x^2-y^2}$ we get

$$x(1 - x^2 + y^2) = 0. \tag{1}$$

Similarly
$$y(1 + x^2 - y^2) = 0. \tag{2}$$
All solutions (x, y) of the system (1), (2) are $(0, 0), (0, 1), (0, -1), (1, 0)$ and $(-1, 0)$. One has $f(1, 0) = f(-1, 0) = e^{-1}$ and f has global maximum at the points $(1, 0)$ and $(-1, 0)$. One has $f(0, 1) = f(0, -1) = -e^{-1}$ and f has global minimum at the points $(0, 1)$ and $(0, -1)$. The point $(0, 0)$ is not an extrema point because of $f(x, 0) = x^2e^{-x^2} > 0$ if $x \neq 0$ and $f(y, 0) = -y^2e^{-y^2} < 0$ if $y \neq 0$.

Problem 3. Set $g(x) = (f(x) + f'(x) + \cdots + f^{(n)}(x))e^{-x}$. From the assumption one get $g(a) = g(b)$. Then there exists $c \in (a, b)$ such that $g'(c) = 0$. Replacing in the last equality $g'(x) = (f^{(n+1)}(x) - f(x))e^{-x}$ we finish the proof.

Problem 4. Set $A = (a_{ij})_{i,j=1}^n, B = (b_{ij})_{i,j=1}^n, AB = (x_{ij})_{i,j=1}^n$ and $BA = (y_{ij})_{i,j=1}^n$. Then $x_{ij} = a_{ii}b_{ij}$ and $y_{ij} = a_{jj}b_{ij}$. Thus $AB = BA$ is equivalent to $(a_{ii} - a_{jj})b_{ij}$ for $i, j = 1, 2, \dots, n$. Therefore $b_{ij} = 0$ if $a_{ii} \neq a_{jj}$ and b_{ij} may be arbitrary if $a_{ii} = a_{jj}$. The number of indices (i, j) for which $a_{ii} = a_{jj} = c_m$ for some $m = 1, 2, \dots, k$ is d_m^2. This gives the desired result.

Problem 5. We define π inductively. Set $\pi(1) = 1$. Assume π is defined for $i = 1, 2, \dots, n$ and also
$$\| \sum_{i=1}^{n} x_{\pi(i)} \|^2 \leq \sum_{i=1}^{n} \| x_{\pi(i)} \|^2 . \tag{1}$$
Note (1) is true for $n = 1$. We choose $\pi(n+1)$ in a way that (1) is fulfilled with $n+1$ instead of n. Set $y = \sum\limits_{i=1}^{n} x_{\pi(i)}$ and $A = \{1, 2, \dots, k\} \setminus \{\pi(i) : i = 1, 2, \dots, n\}$. Assume that $(y, x_r) > 0$ for all $r \in A$. Then $\left(y, \sum\limits_{r\in A} x_r\right) > 0$ and in view of $y + \sum\limits_{r\in A} x_r = 0$ one gets $-(y, y) > 0$, which is impossible. Therefore there is $r \in A$ such that
$$(y, x_r) \leq 0. \tag{2}$$

Put $\pi(n+1)=r$. Then using (2) and (1) we have

$$\| \sum_{i=1}^{n+1} x_{\pi(i)} \|^2 = \| y + x_r \|^2 = \| y \|^2 + 2(y, x_r) + \| x_r \|^2 \le \| y \|^2 + \| x_r \|^2 \le$$

$$\le \sum_{i=1}^{n} \| x_{\pi(i)} \|^2 + \| x_r \|^2 = \sum_{i=1}^{n+1} \| x_{\pi(i)} \|^2,$$

which verifies (1) for $n+1$. Thus we define π for every $n = 1, 2, \dots, k$. Finally from (1) we get

$$\| \sum_{i=1}^{n} x_{\pi(i)} \|^2 \le \sum_{i=1}^{n} \| x_{\pi(i)} \|^2 \le \sum_{i=1}^{k} \| x_i \|^2 .$$

Problem 6. Obviously

$$A_N = \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k . \ln (N-k)} \ge \frac{\ln^2 N}{N} . \frac{N-3}{\ln^2 N} = 1 - \frac{3}{N}. \qquad (1)$$

Take $M, 2 \le M < \dfrac{N}{2}$. Then using that $\dfrac{1}{\ln k . \ln (N-k)}$ is decreasing in $[2, \frac{N}{2}]$ and the symmetry with respect to $\frac{N}{2}$ one get

$$A_N = \frac{\ln^2 N}{N} \Big\{ \sum_{k=2}^{M} + \sum_{k=M+1}^{N-M-1} + \sum_{k=N-M} N-2 \Big\} \frac{1}{\ln k . \ln (N-k)} \le$$

$$\le \frac{\ln^2 N}{N} \Big(2 \frac{M-1}{\ln 2 . \ln (N-2)} + \frac{N-2M-1}{\ln M . \ln (N-M)} \Big\} \le$$

$$\le \frac{2}{\ln 2} . \frac{M \ln N}{N} + \Big(1 - \frac{2M}{N} \Big) \frac{\ln N}{\ln M} + O\Big(\frac{1}{\ln N} \Big)$$

Choose $M = \Big[\dfrac{N}{\ln^2 N} \Big] + 1$ to get

$$A_N \le \Big(1 - \frac{2}{N \ln^2 N} \Big) \frac{\ln N}{\ln N - 2 \ln \ln N} + O\Big(\frac{1}{\ln N} \Big) \le 1 + O\Big(\frac{\ln \ln N}{\ln N} \Big). \quad (2)$$

Estimates (1) and (2) give

$$\lim_{N \to \infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k . \ln (N-k)} = 1.$$

2.2 Solutions of Olympic 1995

2.2.1 Day 1

Problem 1. Let $J = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = 1$ if $i = j+1$ and $a_{ij} = 0$ otherwise. The rank of J is $n-1$ and its only eigenvalues are 0's. Moreover $Y = XJ$ and $A = YX^{-1} = XJX^{-l}, B = X^{-1}Y = J$. It follows that both A and B have rank $n-1$ with only 0's for eigenvalues.
Problem 2. From the inequality

$$0 \leq \int\limits_0^1 (f(x)-x)^2 dx = \int\limits_0^1 f^2(x)dx - 2\int\limits_0^1 xf(x)dx + \int\limits_0^1 x^2 dx$$

we get

$$\int\limits_0^1 f^2(x)dx \geq 2\int\limits_0^1 xf(x)dx - \int\limits_0^1 x^2dx = 2\int\limits_0^1 xf(x)dx - \frac{1}{3}.$$

From the hypotheses we have $\int\limits_0^1 \int\limits_x^1 f(t)dtdx \geq \int\limits_0^1 \frac{1-x^2}{2}dx$ or $\int\limits_0^1 tf(t)dt \geq \frac{1}{3}$. This completes the proof.
Problem 3. Since f' tends to $-\infty$ and f" tends to $+\infty$ as x tends to $0+$, there exists an interval $(0, r)$ such that $f'(x) < 0$ and $f"(x) > 0$ for all $x \in (0, r)$. Hence f is decreasing and f' is increasing on $(0, r)$. By the mean value theorem for every $0 < x < x_0 < r$ we obtain

$$f(x) - f(x_0) = f'(\xi)(x - x_0) > 0,$$

for some $\xi \in (x, x_0)$. Taking into account that f' is increasing, $f'(x) < f'(\xi) < 0$, we get

$$x - x_0 < \frac{f'(\xi)}{f'(x)}(x - x_0) = \frac{f(x) - f(x_0)}{f'(x)} < 0.$$

Taking limits as x tends to $0+$ we obtain

$$-x_0 \leq \lim_{x\to 0+} \inf \frac{f(x)}{f'(x)} \leq \lim_{x\to 0+} \sup \frac{f(x)}{f'(x)} \leq 0.$$

Since this happens for all $x_0 \in (0,r)$ we deduce that $\lim_{x\to 0+} \dfrac{f(x)}{f'(x)}$ exists and $\lim_{x\to 0+} \dfrac{f(x)}{f'(x)} = 0$.

Problem 4. From the definition we have

$$F'(x) = \frac{x-1}{\ln x},\ x > 1.$$

Therefore $F'(x) > 0$ for $x \in (1,\infty)$. Thus F is strictly increasing and hence one-to-one. Since

$$F(x) \geq (x^2 - x)\min\{\frac{1}{\ln t} : x \leq t \leq x^2\} = \frac{x^2-x}{\ln x^2} \to \infty$$

as $x \to \infty$, it follows that the range of F is $(F(1+),\infty)$. In order to determine $F(l+)$ we substitute $t = e^v$ in the definition of F and we get

$$F(x) = \int\limits_{\ln x}^{2\ln x} \frac{e^v}{v} dv.$$

Hence

$$F(x) < e^{2lnx} \int\limits_{\ln x}^{2\ln x} \frac{1}{v} dv = x^2 ln2$$

and similarly $F(x) > x$
$ln2$. Thus $F(1+) = \ln 2$.

Problem 5. We have that

$$(A+tB)^n = A^n + tP_1 + t^2P_2 + \cdots + t^{n-1}P_{n-1} + t^nB^n$$

for some matrices $P_1, P_2, \ldots, P_{n-l}$ not depending on t.

Assume that $a, p_1, p_2, \ldots, p_{n-1}, b$ are the (i,j)-th entries of the corresponding matrices $A^n, P_1, P_2, \ldots, P_{n-1}, B^n$. Then the polynomial

$$bt^n + p_{n-1}t^{n-1} + \cdots + p_2t^2 + p_1t + a$$

has at least $n+1$ roots $t_1, t_2, \ldots, t_{n+1}$. Hence all its coefficients vanish. Therefore $A^n = 0, B^n = 0, P_i = 0$; and A and B are nilpotent.

Problem 6. Let $0 < \delta < 1$. First we show that there exists $K_{p,\delta} > 0$ such that

$$f(x,y) = \frac{(x-y)^2}{4-(x+y)^2} \le K_{p,\delta}$$

for every $(x,y) \in D_\delta = \{(x,y) : |x-y| \ge \delta, |x|^p + |y|^p = 2\}$.

Since D_δ is compact it is enough to show that f is continuous on D_δ. For this we show that the denominator of f is different from zero. Assume the contrary. Then $|x+y| = 2$ and $\left|\frac{x+y}{2}\right|^p = 1$. Since $p > 1$, the function $g(t) = |t|^p$ is strictly convex, in other words $\left|\frac{x+y}{2}\right|^p < \frac{|x|^p + |y|^p}{2}$ whenever $x \ne y$. So for some $(x,y) \in D_\delta$ we have $\left|\frac{x+y}{2}\right|^p < \frac{|x|^p + |y|^p}{2} = 1 = \left|\frac{x+y}{2}\right|^p$. We get a contradiction.

If x and y have different signs then $(x,y) \in D_\delta$ for all $0 < \delta < 1$ because then $|x-y| \ge \max\{|x|,|y|\} \ge 1 > \delta$. So we may further assume without loss of generality that $x > 0, y > 0$ and $x^p + y^p = 2$. Set $x = 1+t$. Then

$$\begin{aligned}
y &= (2-x^p)^{1/p} = (2-(1+t)^p)^{1/p} \\
&= \Big(2 - (1 + pt + \frac{p(p-1)}{2}t^2 + o(t^2))\Big)^{1/p} = (1 - pt - \frac{p(p-1)}{2}t^2 + o(t^2))^{1/p} \\
&= 1 + \frac{1}{p}\Big(-pt - \frac{p(p-1)}{2}t^2 + o(t^2)\Big) + \frac{1}{2p}\Big(\frac{1}{p} - 1\Big)(-pt + o(t^2))^2 + o(t^2) \\
&= 1 - t - \frac{p-1}{2}t^2 + o(t^2) - \frac{p-1}{2}t^2 + o(t^2) \\
&= 1 - t - (p-1)t^2 + o(t^2).
\end{aligned}$$

We have

$$(x-y)^2 = (2t + o(t))^2 = 4t^2 + o(t^2)$$

and

$$\begin{aligned}
4-(x+y)^2 &= 4 - (2 - (p-1)t^2 + o(t^2))^2 \\
&= 4 - 4 + 4(p-1)t^2 + o(t^2) = 4(p-1)t^2 + o(t^2).
\end{aligned}$$

So there exists $\delta_p > 0$ such that if $|t| < \delta_p$ we have $(x-y)^2 < 5t^2, 4-$

$(x+y)^2 > 3(p-1)t^2$. Then

$$(x-y)^2 < 5t^2 = \frac{5}{3(p-1)}.3(p-1)t^2 < \frac{5}{3(p-1)}(4-(x+y)^2) \qquad (*)$$

if $|x-1| < \delta_p$. From the symmetry we have that (*) also holds when $|y-1| < \delta_p$.

To finish the proof it is enough to show that $|x-y| \geq 2\delta_p$ whenever $|x-1| \geq \delta_p, |y-1| \geq \delta_p$ and $x^p+y^p = 2$. Indeed, since $x^p+y^p = 2$ we have that $\max\{x,y\} \geq 1$. So let $x-1 \geq \delta_p$. Since $\left(\frac{x+y}{2}\right)^p \leq \frac{x^p+y^p}{2} = 1$ we get $x+y \leq 2$. Then $x-y \geq 2(x-1) \geq 2\delta_p$.

2.2.2 Day 2

Problem 1. a) Set $A = (a_{ij}), u = (u_1,u_2,u_3)^T$. If we use the orthogonallity condition

$$(Au,u) = 0 \qquad (1)$$

with $u_i = \delta_{ik}$ we get $a_{kk} = 0$. If we use (1) with $u_i = \delta_{ik} + \delta_{im}$ we get

$$a_{kk} + a_{km} + a_{mk} + a_{mm} = 0$$

and hence $a_{km} = -a_{mk}$.

b) Set $v_1 = -a_{23}, v_2 = a_{13}, v_3 = -a_{12}$. Then

$$Au = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1)^T = v \times u.$$

Problem 2. (15 points)

Let $\{b_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that $b_0 = 1, b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1+\sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

Solution. Put $a_n = 1+\sqrt{b_n}$ for $n \geq 0$. Then $a_n > 1, a_0 = 2$ and

$$a_n = 1 + \sqrt{1 + a_{n-1} - 2\sqrt{a_{n-1}}} = \sqrt{a_{n-1}},$$

so $a_n = 2^{2-n}$. Then

$$\begin{aligned}\sum_{n=1}^{N} b_n 2^n &= \sum_{n=1}^{N} (a_n - 1)^2 2^n = \sum_{n=1}^{N} [a_n^2 2^n - a_n 2^{n+1} + 2^n] \\ &= \sum_{n=1}^{N} [(a_{n-1} - 1)2^n - (a_n - 1)2^{n+1}] \\ &= (a_0 - 1)2^1 - (a_N - 1)2^{N+1} = 2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}.\end{aligned}$$

Put $x = 2^{-N}$. Then $x \to 0$ as $N \to \infty$ and so

$$\sum_{n=1}^{\infty} b_n 2^N = \lim_{N\to\infty} \left(2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}\right) = \lim_{x\to 0} \left(2 - 2\frac{2^x - 1}{x}\right) = 2 - 2ln2.$$

Problem 3. It is enough to consider only polynomials with leading coefficient 1. Let $P(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)$ with $|\alpha_j| = 1$, where the complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ may coincide.

We have

$$\begin{aligned}&\widetilde{P}(z) \equiv 2zP'(z) - nP(z) = (z + \alpha_1)(z - \alpha_2) \dots (z - \alpha_n) + \\ &+ (z - \alpha_1)(z + \alpha_2) \dots (z - \alpha_n) + \dots + (z - \alpha_1)(z - \alpha_2) \dots (z + \alpha_n).\end{aligned}$$

Hence, $\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{z + \alpha_k}{z - \alpha_k}$. Since $Re\dfrac{z + \alpha}{z - \alpha} = \dfrac{|z|^2 - |\alpha|^2}{|z - \alpha|^2}$ for all complex $z, \alpha, z \neq \alpha$, we deduce that in our case $Re\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{|z|^2 - 1}{|z - \alpha_k|^2}$. From $|z| \neq 1$ it follows that $Re\dfrac{\widetilde{P}(z)}{P(z)} \neq 0$. Hence $\widetilde{P}(z) = 0$ implies $|z| = 1$.

Problem 4. a) Let n be such that $(1 - \epsilon^2)^n \leq \epsilon$. Then $|x(1 - x^2)^n < \epsilon$ for every $x \in [-1, 1]$. Thus one can set $\lambda_k = (-1)^{k+1}\binom{n}{k}$ because then

$$x - \sum_{k=1}^{n} \lambda_k x^{2k+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k+1} = x(1 - x^2)^n.$$

b) From the Weierstrass theorem there is a polynomial, say $p \in \prod_m$ such that

$$\max_{x\in[-1,1]} |f(x) - p(x)| < \frac{\epsilon}{2}.$$

Set $q(x) = \dfrac{1}{2}\{p(x) - p(-x)\}$. Then

$$f(x) - q(x) = \frac{1}{2}\{f(x) - p(x)\} - \frac{1}{2}\{f(-x) - p(-x)\}$$

and

$$\max_{|x|\le 1} |f(x) - q(x)| \le \frac{1}{2}\max_{|x|\le 1}|f(x) - p(x)| + \frac{1}{2}\max_{|x|\le 1}|f(-x) - p(-x)| < \frac{\epsilon}{2}. \tag{1}$$

But q is an odd polynomial in $\prod_m$ and it can be written as

$$q(x) = \sum_{k=0}^{m} b_k x^{2k+1} = b_0 x + \sum_{k=1}^{m} b_k x^{2k+1}.$$

If $b_0 = 0$ then (1) proves b). If $b_0 \ne 0$ then one applies a) with $\dfrac{\epsilon}{2|b_0|}$ of ϵ to get

$$\max_{|x|\le 1}\left| b_0 x - \sum_{k=1}^{n} b_0 \lambda_k x^{2k+1}\right| < \frac{\epsilon}{2} \tag{2}$$

for appropriate n and $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now b) follows from (1) and (2) with $\max\{n, m\}$ instead of n.

Problem 5. a) Let us consider the integral

$$\int_0^{2\pi} f(x)(1 \pm \cos x)dx = \pi(a_0 \pm 1).$$

The assumption that $f(x) \ge 0$ implies $a_0 \ge 1$. Similarly, if $f(x) \le 0$ then $a_0 \le -1$. In both cases we have a contradiction with the hypothesis of the problem.

b) We shall prove that for each integer N and for each real number $h \ge 24$ and each real number y the function

$$F_N(x) = \sum_{n=1}^{N} \cos(x n^{\frac{3}{2}})$$

changes sign in the interval $(y, y+h)$. The assertion will follow immediately from here.

Consider the integrals

$$I_1 = \int_y^{y+h} F_N(x)dx, \quad I_2 = \int_y^{y+h} F_N(x)\cos x dx.$$

If $F_N(x)$ does not change sign in $(y, y+h)$ then we have

$$|I_2| \leq \int_y^{y+h} |F_N(x)|dx = \Big|\int_y^{y+h} F_N(x)dx\Big| = |I_1|.$$

Hence,it is enough to prove that

$$|I_2| > |I_1|.$$

Obviously, for each $\alpha \neq 0$ we have

$$\Big|\int_y^{y+h} \cos(\alpha x)dx\Big| \leq \frac{2}{|\alpha|}.$$

Hence

$$|I_1| = \Big|\sum_{n=1}^{N} \int_y^{y+h} \cos(xn^{\frac{3}{2}}dx\Big| \leq 2\sum_{n=1}^{N} \frac{1}{n^{\frac{3}{2}}} < 2\Big(1 + \int_1^{\infty} \frac{dt}{t^{\frac{3}{2}}}\Big) = 6. \qquad (1)$$

On the other hand we have

$$\begin{aligned} I_2 &= \sum_{n=1}^{N} \int_y^{y+h} \cos x \cos(xn^{\frac{3}{2}})dx \\ &= \frac{1}{2}\int_y^{y+h} (1+\cos(2x))dx+ \\ &\quad + \frac{1}{2}\sum_{n=2}^{N} \int_y^{y+h} (\cos(x(n^{\frac{3}{2}}-1)) + \cos(x(n^{\frac{3}{2}}+1)))dx \\ &= \frac{1}{2}h + \triangle, \end{aligned}$$

where

$$|\triangle| \le \frac{1}{2}\Big(1 + 2\sum_{n=2}^{N}\Big(\frac{1}{n^{\frac{3}{2}}-1} + \frac{1}{n^{\frac{3}{2}}+1}\Big)\Big) \le \frac{1}{2} + 2\sum_{n=2}^{N}\frac{1}{n^{\frac{3}{2}}-1}.$$

We use that $n^{\frac{3}{2}} - 1 \ge \frac{2}{3}n^{\frac{3}{2}}$ for $n \ge 3$ and we get

$$|\triangle| \le \frac{1}{2} + \frac{2}{2^{\frac{3}{2}}-1} + 3\sum_{n=3}^{N}\frac{1}{n^{\frac{3}{2}}} < \frac{1}{2} + \frac{2}{2\sqrt{2}-1} + 3\int_{2}^{\infty}\frac{dt}{t^{\frac{3}{2}}} < 6.$$

Hence

$$|I_2| > \frac{1}{2}h - 6. \tag{2}$$

We use that $h \ge 24$ and inequalities (1), (2) and we obtain $|I_2| > |I_1|$. The proof is completed.

Problem 6. It is clear that one can add some functions, say $\{g_m\}$, which satisfy the hypothesis of the problem and the closure of the finite linear combinations of $\{f_n\} \cup \{g_m\}$ is $L_2[0,1]$. Therefore without loss of generality we assume that $\{f_n\}$ generates $L_2[0,1]$.

Let us suppose that there is a subsequence $\{n_k\}$ and a function f such that

$$f_{n_k}(x) \underset{k\to\infty}{\longrightarrow} f(x) \text{ for every } x \in [0,1].$$

Fix $m \in \mathbb{N}$. From Lebesgue's theorem we have

$$0 = \int_0^1 f_m(x) f_{n_k}(x)dx \underset{k\to\infty}{\longrightarrow} \int_0^1 f_m(x)f(x)dx.$$

Hence $\int_0^1 f_m(x)f(x)dx = 0$ for every $m \in \mathbb{N}$, which implies $f(x) = 0$ almost everywhere. Using once more Lebesgue's theorem we get

$$1 = \int_0^1 f_{n_k}^2(x)dx \underset{k\to\infty}{\longrightarrow} \int_0^1 f^2(x)dx = 0.$$

The contradiction proves the statement.

2.3 Solutions of Olympic 1996

2.3.1 Day 1

Problem 1. Adding the first column of A to the last column we get that

$$det(A) = (a_0 + a_n)det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ a_1 & a_0 & a_1 & \dots & 1 \\ a_2 & a_1 & a_0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & 1 \end{pmatrix}$$

Subtracting the n-th row of the above matrix from the $(n+1)$-st one, $(n-1)$-st from n-th,..., first from second we obtain that

$$det(A) = (a_0 + a_n)det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ d & -d & -d & \dots & 0 \\ d & d & -d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & 0 \end{pmatrix}$$

Hence,

$$det(A) = (-1)^n (a_0 + a_n)det \begin{pmatrix} d & -d & -d & \dots & -d \\ d & d & -d & \dots & -d \\ d & d & d & \dots & -d \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix}$$

Adding the last row of the above matrix to the other rows we have

$$det(A) = (-1)^n (a_0 + a_n)det \begin{pmatrix} 2d & 0 & 0 & \dots & 0 \\ 2d & 2d & 0 & \dots & 0 \\ 2d & 2d & 2d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix} = (-1)^n (a_0 + a_n) 2^{n-1} d^n.$$

Problem 2. We have

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx \\ &= \int_{0}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx + \int_{-\pi}^{0} \frac{\sin nx}{(1+2^x)\sin x} dx. \end{aligned}$$

In the second integral we make the change of variable $x = -x$ and obtain

$$\begin{aligned} I_n &= \int\limits_0^\pi \frac{\sin nx}{(1+2^x)\sin x}dx + \int\limits_0^\pi \frac{\sin nx}{(1+2^{-x})\sin x}dx \\ &= \int\limits_0^\pi \frac{(1+2^x)\sin nx}{(1+2^x)\sin x}dx \\ &= \int\limits_0^\pi \frac{\sin nx}{\sin x}dx. \end{aligned}$$

For $n \geq 2$ we have

$$\begin{aligned} I_n - I_{n-2} &= \int\limits_0^\pi \frac{\sin nx - \sin(n-2)x}{\sin x}dx \\ &= 2\int\limits_0^\pi \cos(n-1)xdx = 0. \end{aligned}$$

The answer

$$I_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pi & \text{if } n \text{ is odd} \end{cases}$$

follows from the above formula and $I_0 = 0,\ I_1 = \pi$.

Problem 3.

(i) Let $B = \frac{1}{2}(A+E)$. Then

$$B^2 = \frac{1}{4}(A^2 + 2AE + E) = \frac{1}{4}(2AE + 2E) = \frac{1}{2}(A+E) = B.$$

Hence B is a projection. Thus there exists a basis of eigenvectors for B, and the matrix of B in this basis is of the form $diag(1, \dots, 1, 0 \dots, 0)$.

Since $A = 2B - E$ the eigenvalues of A are ± 1 only.

(ii) Let $\{A_i : i \in I\}$ be a set of commuting diagonalizable operators on V, and let A_1 be one of these operators. Choose an eigenvalue λ of A_1 and denote $V_\lambda = \{v \in V : A_1 v = \lambda v\}$. Then V_λ is a subspace of V, and since $A_1 A_i = A_i A_1$ for each $i \in I$ we obtain that V_λ is invariant under each A_i. If $V_\lambda = V$ then A_1 is either E or $-E$, and we can start with another operator A_i. If $V_\lambda \neq V$ we proceed by induction on $dimV$ in order to find a common eigenvector for all A_i. Therefore $\{A_i : i \in I\}$ are simultaneously diagonalizable.

If they are involutions then $|I| \leq 2^n$ since the diagonal entries may equal 1 or -1 only.

Problem 4.

(i) We show by induction that

$$a_n \leq q^n \text{ for } n \geq 3, \tag{*}$$

where $q = 0.7$ and use that $0.7 < 2^{-1/2}$. One has

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, a_4 = \frac{11}{48}.$$

Therefore (*) is true for $n = 3$ and $n = 4$. Assume (*) is true for $n \leq N-1$ for some $N \geq 5$. Then

$$\begin{aligned} \alpha_N &= \frac{2}{N}a_{N-1} + \frac{1}{N}a_{N-2} + \frac{1}{N}\sum_{k=3}^{N-3} a_k a_{N-k} \\ &\leq \frac{2}{N}q^{N-1} + \frac{1}{N}q^{N-2} + \frac{N-5}{N}q^N \leq q^N \end{aligned}$$

because $\dfrac{2}{q} + \dfrac{1}{q^2} \leq 5.$

ii) We show by induction that

$$a_n \geq q^n \text{ for } n \geq 2,$$

where $q = \dfrac{2}{3}$. One has $a_2 = \dfrac{1}{2} > \left(\dfrac{2}{3}\right)^2 = q^2$. Going by induction we have for $N \geq 3$.

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}\sum_{k=2}^{N-2} a_k a_{N-k} \geq \frac{2}{N}q^{N-1} + \frac{N-2}{N}q^N = q^N$$

because $\dfrac{2}{q} = 3.$

Problem 5. i) With a linear change of the variable (i) is equivalent to:

i') Let a, b, A be real numbers such that $b \le 0, A > 0$ and $1+ax+bx^2 > 0$ for every x in $[0, A]$. Denote $I_n = n\int\limits_0^A (1+ax+bx^2)^n dx$. Prove that $\lim_{n\to+\infty} I_n = -\dfrac{1}{a}$ when $a < 0$ and $\lim_{n\to+\infty} I_n = +\infty$ when $a \ge 0$.

Let $a < 0$. Set $f(x) = e^{ax} - (1+ax+bx^2)$. Using that $f(0) = f'(0) = 0$ and $f''(x) = a^2e^{ax} - 2b$ we get for $x > 0$ that

$$0 < e^{ax} - (1+ax+bx^2) < cx^2$$

where $c = \dfrac{a^2}{2} - b$. Using the mean value theorem we get

$$0 < e^{anx} - (1+ax+bx^2)^n < cx^2ne^{a(n-1)x}.$$

Therefore

$$0 < n\int\limits_0^A e^{anx}dx - n\int\limits_0^A (1+ax+bx^2)^n dx < cn^2\int\limits_0^A x^2e^{a(n-1)x}dx.$$

Using that

$$n\int\limits_0^A e^{anx}dx = \frac{e^{anA}-1}{a} \underset{n\to\infty}{\longrightarrow} -\frac{1}{a}$$

and

$$\int\limits_0^A x^2e^{a(n-1)x}dx < \frac{1}{|a|^3(n-1)^3}\int\limits_0^\infty t^2e^{-t}dt$$

we get (i') in the case $a < 0$.

Let $a \ge 0$. Then for $n > \max\{A^{-2}, -b\} - 1$ we have

$$n\int\limits_0^A (1+ax+bx^2)^n dx > n\int\limits_0^{\frac{1}{\sqrt{n+1}}} (1+bx^2)^n dx$$

$$n \cdot \frac{1}{\sqrt{n+1}} \cdot \left(1+\frac{b}{n+1}\right)^n$$

$$\frac{n}{\sqrt{n+1}}e^b \underset{n\to\infty}{\longrightarrow} \infty.$$

(i) is proved.

ii) Denote $I_n = n \int\limits_0^1 (f(x))^n dx$ and $M = \max\limits_{x \in [0,1]} f(x)$.

For $M < 1$ we have $I_n \le nM^n \underset{n\to\infty}{\longrightarrow} 0$, a contradiction.

If $M > 1$ since f is continuous there exists an interval $I \subset [0,1]$ with $|I| > 0$ such that $f(x) > 1$ for every $x \in I$. Then $I_n \ge n|I| \underset{n\to\infty}{\longrightarrow} +\infty$, a contradiction. Hence $M = 1$. Now we prove that f' has a constant sign. Assume the opposite. Then $f'(x_0) = 0$ for some $x \in (0,1)$. Then $f(x_0) = M = 1$ because $f'' \le 0$. For $x_0 + h$ in $[0,1]$, $f(x_0 + h) = 1 + \frac{h^2}{2} f''(\xi)$, $\xi \in (x_0, x_0 + h)$. Let $m = \min\limits_{x \in [0,1]} f''(x)$. So, $f(x_0 + h) \ge 1 + \frac{h^2}{2} m$.

Let $\delta > 0$ be such that $1 + \frac{\delta^2}{2} m > 0$ and $x_0 + \delta < 1$. Then

$$I_n \ge n \int\limits_{x_0}^{x_0+\delta} (f(x))^n dx \ge n \int\limits_0^{\delta} \left(1 + \frac{m}{2} h^2\right)^n dh \underset{n\to\infty}{\longrightarrow} \infty$$

in view of (i')-a contradiction. Hence f is monotone and $M = f(0)$ or $M = f(1)$.

Let $M = f(0) = 1$. For h in $[0,1]$

$$1 + hf'(0) \ge f(h) \ge 1 + hf'(0) + \frac{m}{2} h^2,$$

where $f'(0) \ne 0$, because otherwise we get a contradiction as above. Since $f(0) = M$ the function f is decreasing and hence $f'(0) < 0$. Let $0 < A < 1$ be such that $1 + Af'(0) + \frac{m}{2} A^2 > 0$. Then

$$n \int\limits_0^A (1 + hf'(0))ndh \ge n \int\limits_0^A (f(x))^n dx \ge n \int\limits_0^A (1 + hf'(0) + \frac{m}{2} h^2)^n dh.$$

From (i') the first and the third integral tend to $-\frac{1}{f'(0)}$ as $n \to \infty$, hence so does the second.

Also $n \int\limits_A^1 (f(x))^n dx \le n(f(A))^n \underset{n\to\infty}{\longrightarrow} 0$ $(f(A) < 1)$. We ger $L = -\frac{1}{f'(0)}$ in this case.

If $M = f(1)$ we get in a similar way $L = \dfrac{1}{f'(1)}$.

Problem 6.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2,2),(2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks: All distances used in this problem are Euclidian. Diameter of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x,y) : x,y \in E\}$. Contraction of a set E to a set F is a mapping $f : E \mapsto F$ such that $\operatorname{dist}(f(x),f(y)) \le \operatorname{dist}(x,y)$ for all $x,y \in E$. A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. Triangle is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

Solution.

(a) The choice $E_1 = L$ gives $\mathcal{C}(L) \le lenght(L)$. If $E \supset \bigcup\limits_{i=1}^{n} E_i$ then $\sum\limits_{i=1}^{n} \operatorname{diam}(E_i) \ge \operatorname{lenght}(L)$: By induction, $n = l$ obvious, and assuming that E_{n+1} contains the end point a of L, define the segment $L_\epsilon = \{x \in L : \operatorname{dist}(x,a) \ge \operatorname{diam}(E_{n+1}) + \epsilon\}$ and use induction assumption to get $\sum\limits_{i=1}^{n+1} \operatorname{diam}(E_i) \ge \operatorname{lenght}(L_\epsilon) + \operatorname{diam}(E_{n+1}) \ge \operatorname{lenght}(L) - \epsilon$; but $\epsilon > 0$ is arbitrary.

(b) If f is a contraction of E onto L and $E \subset \bigcup\limits_{i=1}^{n} E_i$ then $L \subset \bigcup\limits_{i=1}^{n} f(E_i)$ and $\operatorname{lenght}(L) \le \sum\limits_{i=1}^{n} \operatorname{diam}(f(Ei)) \le \sum\limits_{i=1}^{n} \operatorname{diam}(Ei)$.

(c1) Let $E = T \cup T'$ where T is the triangle with vertices $(-2,2),(2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis. Suppose $E \subset \bigcup\limits_{i=1}^{n} E_i$. If no set among E_i meets both T and T', then E_i may be partitioned into covers of segments $[(-2,2),(2,2)]$ and $[(-2,2),(2,-2)]$, both of length 4, so $\sum\limits_{i=1}^{n} \operatorname{diam}(E_i) \ge 8$. If at least one set among E_i, say E_k, meets both T and T', choose $a \in E_k \cap T$ and $b \in E_k \cap T'$ and note that the sets $E'_i = E_i$ for $i \ne k, E'_k = E_k \cup [a,b]$ cover $T \cup T' \cup [a,b]$, which is a set of upper content at least 8, since its orthogonal projection onto y-axis is a segment of length 8. Since diam (E_j) = diam (E'_j), we get $\sum_{i=1}^{n}$ diam $(E_i) \geq 8$.

(c2) Let f be a contraction of E onto $L = [a', b']$. Choose $a = (a_l, a_2), b = (b_1, b_2) \in E$ such that $f(a) = a'$ and $f(b) = b'$. Since lenght (L) = dist (a', b') $\leq$ dist (a, b) and since the triangles have diameter only 4, we may assume that $a \in T$ and $b \in T'$. Observe that if $a_2 \leq 3$ then a lies on one of the segments joining some of the points $(-2, 2), (2, 2), (-1, 3), (1, 3)$; since all these points have distances from vertices, and so from points, of T_2 at most $\sqrt{50}$, we get that lenght $(L) \leq$ dist $(a, b) \leq \sqrt{50}$. Similarly if $b_2 \geq -3$. Finally, if $a_2 > 3$ and $b_2 < -3$, we note that every vertex, and so every point of T is in the distance at most $\sqrt{10}$ for a and every vertex, and so every point, of T' is in the distance at most $\sqrt{10}$ of b. Since f is a contraction, the image of T lies in a segment containing a' of length at most $\sqrt{10}$ and the image of T' lies in a segment containing b' of length at most $\sqrt{10}$. Since the union of these two images is L, we get lenght $(L) \leq 2\sqrt{10} \leq \sqrt{50}$. Thus $\mathcal{K}(E) \leq \sqrt{50} < 8$.

2.3.2 Day 2

Problem 1. The "only if" part is obvious. Now suppose that $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$ and the sequence $\{x_n\}$ does not converge. Then there are two cluster points $K < L$. There must be points from the interval (K, L) in the sequence. There is an $x \in (K, L)$ such that $f(x) \neq x$. Put $\epsilon = \frac{|f(x) - x|}{2} > 0$. Then from the continuity of the function f we get that for some $\delta > 0$ for all $y \in (x - \delta, x + \delta)$ it is $|f(y) - y| > \epsilon$. On the other hand for n large enough it is $|x_{n+1} - x_n| < 2\delta$ and $|f(x_n) - x_n| = |x_{n+1} - x_n| < \epsilon$. So the sequence cannot come into the interval $(x - \delta, x + \delta)$, but also cannot jump over this interval. Then all cluster points have to be at most $x - \delta$ (a contradiction with L being a cluster point), or at least $x+\delta$ (a contradiction with $\mathcal{K}$ being a cluster point).

Problem 2. First we show that

$$\text{If } \cosh t \text{ is rational and } m \in \mathbb{N}, \text{ then } \cosh mt \text{ is rational.} \tag{1}$$

Since $\cosh 0.t = \cosh 0 = 1 \in \mathbb{Q}$ and $\cosh 1.t = \cosh t \in \mathbb{Q}$, (1) follows inductively from

$$\cosh(m+1)t = 2\cosh t.coshmt - \cosh(m-1)t.$$

The statement of the problem is obvious for $k=1$, so we consider $k \geq 2$. For any m we have

$$\begin{aligned}\cosh\theta &= \cosh((m+1)\theta - m\theta) = \\ &= \cosh(m+1)\theta.\cosh m\theta - \sinh(m+1)\theta.\sinh m\theta \qquad (2)\\ &= \cosh(m+1)\theta.\cosh m\theta - \sqrt{\cosh^2(m+1)\theta - 1}\sqrt{\cosh^2 m\theta - 1}\end{aligned}$$

Set $\cosh k\theta = a, \cosh(k+1)\theta = b, a, b \in \mathbb{Q}$. Then (2) with $m=k$ gives

$$\cosh\theta = ab - \sqrt{a^2-1}\sqrt{b^2-1}$$

and then

$$\begin{aligned}(a^2-1)(b^2-1) &= (ab - \cosh\theta)^2 \\ &= a^2b^2 - 2ab\cosh\theta + \cosh^2\theta.\end{aligned} \tag{3}$$

Set $\cosh(k^2-1)\theta = A, coshk^2\theta = B$. From (1) with $m=k-1$ and $t=(k+1)\theta$ we have $A \in \mathbb{Q}$. From (1) with $m=k$ and $t=k\theta$ we have $B \in \mathbb{Q}$. Moreover $k^2-1>k$ implies $A>a$ and $B>b$. Thus $AB>ab$. From (2) with $m=k^2-1$ we have

$$\begin{aligned}(A^2-1)(B^2-1) &= (AB - \cosh\theta)^2 \\ &= A^2B^2 - 2AB\cosh\theta + \cosh^2\theta.\end{aligned} \tag{4}$$

So after we cancel the $\cosh^2\theta$ from (3) and (4) we have a non-trivial linear equation in $\cosh\theta$ with rational coefficients.

Problem 3. (a) All of the matrices in G are of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

So all of the matrices in H are of the form

$$M(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

so they commute. Since $M(x)^{-l} = M(-x)$, H is a subgroup of G.

(b) A generator of H can only be of the form $M(x)$, where x is a binary rational, i.e., $x = \dfrac{p}{2^n}$ with integer p and non-negative integer n. In H it holds

$$M(x)M(y) = M(x+y)$$
$$M(x)M(y)^{-1} = M(x-y).$$

The matrices of the form $M(\frac{1}{2^n})$ are in H for all $n \in \mathbb{N}$. With only finite number of generators all of them cannot be achieved.

Problem 4. Assume the contrary - there is an $arcA \subset C$ with length $l(A) = \dfrac{\pi}{2}$ such that $A \subset B \backslash \Gamma$. Without loss of generality we may assume that the ends of A are $M = (\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}), N = (\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}})$. A is compact and Γ is closed. From $A \cap \Gamma = \emptyset$ we get $\delta > 0$ such that $dist(x,y) > \delta$ for every $x \in A, y \in \Gamma$.

Given $\epsilon > 0$ with E_ϵ we denote the ellipse with boundary: $\dfrac{x^2}{(1+\epsilon)^2} + \dfrac{y^2}{b^2} = 1$, such that $M, N \in E_\epsilon$. Since $M \in E_\epsilon$ we get

$$b^2 = \frac{(1+\epsilon)^2}{2(1+\epsilon)^2 - 1}.$$

Then we have

$$areaE_\epsilon = \pi \frac{(1+\epsilon)^2}{\sqrt{2(1+\epsilon)^2 - 1}} > \pi = areaD.$$

In view of the hypotheses, $E_\epsilon \backslash \neq \emptyset$ for every $\epsilon > 0$. Let $S = \{(x,y) \in \mathbb{R}^2 : |x| > |y|\}$. From $E_\epsilon \backslash S \subset D \subset B$ it follows that $E_\epsilon \backslash B \subset S$. Taking $\epsilon < \delta$ we get that

$$\emptyset \neq E_\epsilon \backslash B \subset E_\epsilon \cap S \subset D_{1+\epsilon} \cup S \subset B$$

- a contradiction (we use the notation $D_t = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le t^2\}$).

Remark. The ellipse with maximal area is well known as John's ellipse. Any coincidence with the President of the Jury is accidental.

Problem 5.

(1) Set $f(t) = \dfrac{t}{(1+t^2)^2}$, $h = \dfrac{1}{\sqrt{x}}$. Then

$$\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = h\sum_{n=1}^{\infty} f(nh) \underset{h\to 0}{\to} \int_0^{\infty} f(t)dt = \frac{1}{2}.$$

The convergence holds since $h\sum\limits_{n=1}^{\infty} f(nh)$ is a Riemann sum of the integral $\int\limits_0^{\infty} f(t)dt$. There are no problems with the infinite domain because f is integrable and $f \downarrow 0$ for $x \to \infty$ (thus $h\sum\limits_{n=N}^{\infty} f(nh) \ge \int\limits_{nN}^{\infty} f(t)dt \ge h\sum\limits_{n=N+1}^{\infty} f(nh)$).

(ii) We have

$$\begin{aligned}\Big|\sum_{i=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\Big| &= \Big|\sum_{n=1}^{\infty}\Big(hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\Big) - \int_0^{\frac{h}{2}} f(t)dt\Big| \\ &\le \sum_{n=1}^{\infty}\Big|hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\Big| + \int_0^{\frac{h}{2}} f(t)dt\end{aligned} \tag{1}$$

Using twice integration by parts one has

$$2bg(a) - \int_{a-b}^{a+b} g(t)dt = -\frac{1}{2}\int_0^b (b-t)^2(g''(a+t)+g''(a-t))dt \tag{2}$$

for every $g \in C^2[a-b, a+b]$. Using $f(0)=0, f \in C^2[0,h/2]$ one gets

$$\int_0^{h/2} f(t)dt = O(h^2). \tag{3}$$

From (1), (2) and (3) we get

$$\Big|\sum_{i=1}^{\infty}\frac{nx}{(n^2+x)^2}-\frac{1}{2}\Big|\le\sum_{n=1}^{\infty}h^2\int\limits_{nh-\frac{1}{2}}^{nh+\frac{1}{2}}|f''(t)|dt+O(h^2)=$$

$$=h^2\int\limits_{\frac{1}{2}}^{\infty}|f''(t)|dt+O(h^2)=O(h^2)=O(x^{-1}).$$

Problem 6.

(i) Put for $n\in\mathbb{N}$

$$c_n=\frac{(n+1)^n}{n^{n-1}} \tag{2.1}$$

Observe that $c_1c_2\dots c_n=(n+1)^n$. Hence, for $n\in\mathbb{N}$,

$$(a_1a_2\dots a_n)^{1/n}=\frac{(a_1c_1a_2c_2\dots a_nc_n)^{1/n}}{(n+1)}$$
$$\le\frac{(a_1c_1+\dots+a_nc_n)}{n(n+1)}.$$

Consequently,

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{1/n}\le\sum_{n=1}^{\infty}a_nc_n\Big(\sum_{m=n}^{\infty}(m(m+1))^{-1}\Big). \tag{2}$$

Since

$$\sum_{m=n}^{\infty}(m(m+1))^{-1}=\sum_{m=n}^{\infty}\Big(\frac{1}{m}-\frac{1}{m+1}\Big)=\frac{1}{n}$$

we have

$$\sum_{n=1}^{\infty}a_nc_n\Big(\sum_{m=n}^{\infty}(m(m+1))^{-1}\Big)=\sum_{n=1}^{\infty}\frac{a_nc_n}{n}$$

$$=\sum_{n=1}^{\infty}a_n(\frac{(n+1)}{n})^n<e\sum_{n=1}^{\infty}a_n$$

(by (1)). Combining the last inequality with (2) we get the result.

(ii) Set $a_n=n^{n-1}(n+1)^{-n}$ for $n=1,2,\dots,N$ and $a_n=2^{-n}$ for $n>N$, where N will be chosen later. Then

$$(a_1\dots a_n)^{1/n}=\frac{1}{n+1} \tag{3}$$

for $n \le N$. Let $K = K(\epsilon)$ be such that

$$\left(\frac{n+1}{n}\right)^n > \epsilon - \frac{\epsilon}{2} \text{ for } n > K. \tag{4}$$

Choose N from the condition

$$\sum_{n=1}^{K} a_n + \sum_{n=1}^{\infty} 2^{-n} \le \frac{\epsilon}{(2e-\epsilon)(e-\epsilon)} \sum_{n=K+1}^{N} \frac{1}{n}, \tag{5}$$

which is always possible because the harmonic series diverges. Using (3), (4) and (5) we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{K} a_n + \sum_{n=N+1}^{\infty} 2^{-n} + \sum_{n=K+1}^{N} \frac{1}{n}\left(\frac{n}{n+1}\right)^n <$$

$$< \frac{\epsilon}{(2e-\epsilon)(e-\epsilon)} \sum_{n=K+1}^{N} \frac{1}{n} + \left(e - \frac{\epsilon}{2}\right)^{-1} \sum_{n=K+1}^{N} \frac{1}{n} =$$

$$= \frac{1}{e-\epsilon} \sum_{n=K+1}^{N} \frac{1}{n} \le \frac{1}{e-\epsilon} \sum_{n=1}^{\infty} (a_1 \dots a_n)^{1/n}.$$

2.4 Solutions of Olympic 1997

2.4.1 Day 1

Problem 1.

It is well known that

$$-1 = \int_0^1 \ln x dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right)$$

(Riemman's sums). Then

$$\frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \epsilon_n\right) \ge \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} -1.$$

Given $\epsilon > 0$ there exist n_0 such that $0 < \epsilon_n \le \epsilon$ for all $n \ge n_0$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \epsilon_n\right) \le \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \epsilon\right).$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\epsilon\right)=\int_0^1\ln(x+\epsilon)dx$$
$$=\int_\epsilon^{1+\epsilon}\ln x dx$$

we obtain the result when ϵ goes to 0 and so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\epsilon_n\right)=-1.$$

Problem 2.

a) Yes. Let $S=\sum\limits_{n=1}^{\infty}a_n, S_n=\sum\limits_{k=1}^{n}a_k$. Fix $\epsilon>0$ and a number no such that $|S_n-S|<\epsilon$ for $n>n_0$. The partial sums of the permuted series have the form $L_{2^{n-1}+k}=S_{2^{n-1}}+S_{2^n}-S_{2^n-k}, 0\le k<2^{n-1}$ and for $2^{n-1}>n_0$ we have $|L_{2^{n-1}+k}-S|<3\epsilon$, i.e. the permuted series converges.

b) No. Take $a_n=\dfrac{(-1)^{n+1}}{\sqrt{n}}$. Then $L_{3.2^{n-2}}=S_{2^{n-1}}+\sum\limits_{k=2^{n-2}}^{2^{n-1}-1}\dfrac{1}{\sqrt{2k+1}}$ and $L_{3.2^{n-2}}-S_{2^{n-1}}\ge 2^{n-2}\dfrac{1}{\sqrt{2^n}}\underset{n\to\infty}{\longrightarrow}\infty$, so $L_{3.2^{n-2}}\underset{n\to\infty}{\longrightarrow}\infty$.

Problem 3.

Set $S=A+\omega B$, where $\omega=-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$. We have

$$S\overline{S}=(A+\omega B)(A+\overline{\omega}B)=A^2+\omega BA+\overline{\omega}AB+B^2$$
$$=AB+\omega BA+\overline{\omega}AB=\omega(BA-AB),$$

because $\overline{\omega}+1=-\omega$. Since $det(S\overline{S})=detS.det\overline{S}$ is a real number and $det\omega(BA-AB)=\omega^n$ and $det(BA-AB)\neq 0$, then ω^n is a real number. This is possible only when n is divisible by 3.

Problem 4.

a) We construct inductively the sequence $\{n_i\}$ and the ratios

$$\theta_k=\frac{\alpha}{\prod_1^k(1+\frac{1}{n_i})}$$

so that

$$\theta_k > 1 \text{ for all } k.$$

Choose n_k to be the least n for which

$$n + \frac{1}{n} < \theta_{k-1}$$

$(\theta_0 = \alpha)$ so that for each k,

$$1 + \frac{1}{n_k} < \theta_{k-1} \leq 1 + \frac{1}{n_k - 1}. \tag{1}$$

Since

$$\theta_{k-1} \leq 1 + \frac{1}{n_k - 1}$$

we have

$$1 + \frac{1}{n_{k+1}} < \theta_k = \frac{\theta_{k-1}}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_k - 1}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Hence, for each $k, n_{k+1} \geq n_k^2$.

Since $n_1 \geq 2, n_k \to \infty$ so that $\theta_k \to 1$. Hence

$$\alpha = \prod_1^\infty \Big(1 + \frac{1}{n_k}\Big).$$

The uniquness of the infinite product will follow from the fact that on every step n_k has to be determine by (1).

Indeed, if for some k we have

$$1 + \frac{1}{n_k} \geq \theta_{k-1}$$

then $\theta_k \leq 1, \theta_{k+1} < 1$ and hence $\{\theta_k\}$ does not converge to 1.

Now observe that for $M > 1$,

$$\Big(1+\frac{1}{M}\Big)\Big(1+\frac{1}{M^2}\Big)\Big(1+\frac{1}{M^4}\Big)\cdots = 1+\frac{1}{M}+\frac{1}{M^2}+\frac{1}{M^3}+\cdots = 1+\frac{1}{M-1}. \tag{2}$$

Assume that for some k we have

$$1 + \frac{1}{n_k - 1} < \theta_{k-1}.$$

Then we get

$$\frac{\alpha}{(1+\frac{1}{n_1})(1+\frac{1}{n_2})\cdots} = \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_{k+1}})\cdots}$$
$$\geq \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_k^2})\cdots} = \frac{\theta_{k-1}}{1+\frac{1}{n_k-1}} > 1$$

- a contradiction,

b) From (2) α is rational if its product ends in the stated way.

Conversely, suppose α is the rational number $\frac{p}{q}$, Our aim is to show that for some m,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}.$$

Suppose this is not the case, so that for every m,

$$\theta_{m-1} < \frac{n_m}{n_m - 1}. \qquad (3)$$

For each k we write

$$\theta_k = \frac{p_k}{q_k}$$

as a fraction (not necessarily in lowest terms) where

$$p_0 = p, q_0 = q$$

and in general

$$p_k = p_{k-1}n_k, q_k = q_{k-1}(N_k + 1).$$

The numbers $p_k - q_k$ are positive integers: to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$p_k - q_k - (p_{k-1} - q_{k-1}) = n_k p_{k-1} - (n_k + 1)q_{k-1} - p_{k-1} + q_{k-1}$$
$$= (n_k - 1)p_{k-1} - n_k q_{k-1}$$

and this is negative because

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3).

Problem 5.

a) For $x = a$ the statement is trivial. Let $x \neq 0$. Then $\max_i x_i > 0$ and $\min_i x_i < 0$. Hence $\| x \|_\infty < 1$. From the hypothesis on x it follows that:

i) If $x_j \leq 0$ then $\max_i x_i \leq x_j + 1$.

ii) If $x_j \geq 0$ then $\min_i x_i \geq x_j - 1$.

Consider $y \in Z_0^n, y \neq 0$. We split the indices $\{1, 2, \ldots, n\}$ into five sets:

$$I(0) = \{i : y_i = 0\},$$
$$I(+,+) = \{i : y_i > 0, x_i \geq 0\}, \quad I(+,-) = \{i : y_i > 0, x_i < 0\},$$
$$I(-,+) = \{i : y_i < 0, x_i > 0\}, \quad I(-,-) = \{i : y_i < 0, x_i \leq 0\}$$

As least one of the last four index sets is not empty. If $I(+,+) \neq \emptyset$ or $I(-,-) \neq \emptyset$ then $\| x + y \|_\infty \geq 1 > \| x \|_\infty$. If $I(+,+) = I(-,-) = \emptyset$ then $\sum y_i = 0$ implies $I(+,-) \neq \emptyset$ and $I(-,+) \neq \emptyset$. Therefore i) and ii) give $\| x + y \|_\infty \geq \| x \|_\infty$ which completes the case $p = \infty$.

Now let $1 \leq p < \infty$. Then using i) for every $j \in I(+,-)$ we get $|x_j + y_j| = y_j - 1 + x_j + 1 \geq |y_j| - 1 + \max_i x_i$. Hence

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \text{ for every } k \in I(-,+) \text{ and } j \in I(+,-).$$

Similarly

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \text{ for every } k \in I(+,-) \text{ and } j \in I(-,+);$$
$$|x_j + y_j|^p \geq |y_j| + |x_j|^p \text{ for every } j \in I(+,+) \cup I(-,-).$$

Assume that $\sum\limits_{j\in(+,-)} 1 \geq \sum\limits_{j\in I(-,+)} 1$. Then

$$\begin{aligned}
&\| x+y \|_p^p - \| x \|_p^p \\
&= \sum_{j\in I(+,+)\cup I(-,-)} (|x_j+y_j|^p - |x_j|^p) + \Big(\sum_{j\in I(+,-)} |x_j+y_j|^p - \sum_{k\in I(-,+)} |x_k|^p \Big) \\
&\quad + \Big(\sum_{j\in I(-,+)} |x_j+y_j|^p - \sum_{k\in I(+,-)} |x_k|^p \Big) \\
&\geq \sum_{j\in I(+,+)\cup I(-,-)} |y_j| + \sum_{j\in I(+,-)} (|y_j|-1) \\
&\quad + \Big(\sum_{j\in I(-,+)} (|y_j|-1) - \sum_{j\in I(+,-)} 1 + \sum_{j\in I(-,+)} 1 \Big) \\
&= \sum_{i=1}^{n} |y_i| - 2\sum_{j\in I(+,-)} 1 = 2\sum_{j\in I(+,-)} (y_j-1) + 2\sum_{j\in I(+,+)} y_j \geq 0.
\end{aligned}$$

The case $\sum\limits_{j\in I(+,-)} 1 \leq \sum\limits_{j\in I(-,+)} 1$ is similar. This proves the statement.

b) Fix $p \in (0,1)$ and a rational $t \in (\frac{1}{2}, 1)$. Choose a pair of positive integers m and l such that $mt = l(1-t)$ and set $n = m+l$. Let

$$x_i = t,\ i = 1,2,\dots,m;\quad x_i = t-1,\ i = m+1, m+2, \dots, n;$$
$$y_i = -1,\ i = 1,2,\dots,m;\quad y_{m+1} = m;\ y_i = 0, i = m+2,\dots,n.$$

Then $x \in R_0^n, \max\limits_i x_i - \min\limits_i x_i = 1, y \in Z_0^n$ and

$$\| x \|_p^p - \| x+y \|_p^p = m(t^p - (1-t)^p) + (1-t)^p - (m-1+t)^p,$$

which is possitive for m big enough.

Problem 6.

a) No.

Consider $F = \{A_1, B_1, \dots, A_n, B_n, \dots\}$, where $A_n = \{1,3,5,\dots,2n-1, 2n\}$, $B_n = \{2,4,6,\dots,2n,2n+1\}$.

b) Yes.

We will prove inductively a stronger statement:

Suppose F, G are two families of finite subsets of $\mathbb{N}$ such that:

1) For every $A \in F$ and $B \in G$ we have $A \cap B \neq \emptyset$;

2) All the elements of F have the same size r, and elements of G- size s. (we shall write $\#(F) = r, \#(G) = s$).

Then there is a finite set Y such that $A \cup B \cup Y \neq \emptyset$ for every $A \in F$ and $\in G$.

The problem b) follows if we take $F = G$.

Proof of the statement: The statement is obvious for $r = s = 1$. Fix the numbers r, s and suppose the statement is proved for all pairs F', G' with $\#(F') < r, \#(G') < s$. Fix $A_0 \in F, B_0 \in G$. For any subset $C \subset A_0 \cup B_0$, denote

$$F(C) = \{A \in F : A \cap (A_0 \cup B_0) = C\}.$$

Then $F = \bigcup\limits_{\emptyset \neq C \subset A_0 \cup B_0} F(C)$. It is enough to prove that for any pair of non-empty sets $C, D \subset A_0 \cup B_0$ the families $F(C)$ and $G(D)$ satisfy the statement.

Indeed, if we denote by $Y_{C,D}$ the corresponding finite set, then the finite set $\bigcup\limits_{C,D \subset A_0 \cup B_0} Y_{C,D}$ will satisfy the statement for F and G. The proof for $F(C)$ and $G(D)$.

If $C \cap D \neq \emptyset$, it is trivial.

If $C \cap D = \emptyset$, then any two sets $A \in F(C), B \in G(D)$ must meet outside $A_0 \cup B_0$. Then if we denote $\widetilde{F}(C) = \{A \backslash C : A \in F(C)\}, \widetilde{G}(D) = \{B \backslash D : B \in G(D)\}$, then $\widetilde{F}(C)$ and $\widetilde{G}(D)$ satisfy the conditions 1) and 2) above, with $\#(\widetilde{F}(C)) = \#(F) - \#C < r, \#(\widetilde{G}(D)) = \#(G) - \#D < s$, and the inductive assumption works.

2.4.2 Day 2

Problem 1.

Let $c = \frac{1}{2} f''(0)$. We have

$$g = \frac{(f')^2 - 2ff''}{2(f')^2\sqrt{f}},$$

where

$$f(x) = cx^2 + O(x^3),\ f'(x) = 2cx + O(x^2),\ f''(x) = 2c + O(x).$$

Therefore $(f'(x))^2 = 4c^2x^2 + O(x^3)$,

$$2f(x)f''(x) = 4c^2x^2 + O(x^3)$$

and

$$2(f'(x))^2\sqrt{f(x)} = 2(4c^2x^2 + O(x^3))|x|\sqrt{c + O(x)}.$$

g is bounded because

$$\frac{2(f'(x))^2\sqrt{f(x)}}{|x|^3} \underset{x\to 0}{\longrightarrow} 8c^{5/2} \neq 0$$

and $f'(x)^2 - 2f(x)f"(x) = O(x^3)$.

The theorem does not hold for some C^2-functions.

Let $f(x) = (x + |x|^{3/2})^2 = x^2 + 2x^2\sqrt{|x|} + |x|^3$, so f is C^2. For $x > 0$,

$$g(x) = \frac{1}{2}\Big(\frac{1}{1+\frac{3}{2}\sqrt{x}}\Big)' = -\frac{1}{2}\cdot\frac{1}{(1+\frac{3}{2}\sqrt{x})^2}\cdot\frac{3}{4}\cdot\frac{1}{\sqrt{x}} \underset{x\to 0}{\longrightarrow} -\infty.$$

Problem 2.

Let I denote the identity $n \times n$ matrix. Then

$$detM.detH = det\begin{bmatrix} A & B \\ C & D \end{bmatrix}.det\begin{bmatrix} I & F \\ 0 & H \end{bmatrix} = det\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} = detA.$$

Problem 3.

Set $f(t) = \dfrac{\sin(\log t)}{t^\alpha}$. We have

$$f'(x) = \frac{-\alpha}{t^{\alpha+1}}\sin(\log t) + \frac{\cos(\log t)}{t^{\alpha+1}}.$$

So $|f'(t)| \leq \dfrac{1+\alpha}{t^{\alpha+1}}$ for $\alpha > 0$. Then from Mean value theorem for some $\theta \in (0,1)$ we get $|f(n+1) - f(n)| = |f'(n+\theta)| \leq \dfrac{1+\alpha}{n^{\alpha+1}}$. Since $\sum \dfrac{1+\alpha}{n^{\alpha+1}} < +\infty$ for $\alpha > 0$ and $f(n) \underset{n\to\infty}{\longrightarrow} 0$ we get that $\sum_{n=1}^{\infty}(-1)^{n-1}f(n) = \sum_{n=1}^{\infty}(f(2n-1) - f(2n))$ converges.

Now we have to prove that $\dfrac{\sin(\log n)}{n^{\alpha}}$ does not converge to 0 for $\alpha \le 0$. It suffices to consider $\alpha = 0$.

We show that $a_n = \sin(\log n)$ does not tend to zero. Assume the contrary. There exist $k_n \in \mathbb{N}$ and $\lambda_n \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for $n > e^2$ such that $\dfrac{\log n}{\pi} = k_n + \lambda_n$. Then $|a_n| = \sin \pi|\lambda_n|$. Since $a_n \to 0$ we get $\lambda_n \to 0$. We have

$$k_{n+1} - k_n = \\ = \frac{\log(n+1) - \log n}{\pi} - (\lambda_{n+1} - \lambda_n) = \frac{1}{\pi}\log\left(1 + \frac{1}{n}\right) - (\lambda_{n+1} - \lambda_n).$$

Then $|k_{n+1} - k_n| < 1$ for all n big enough. Hence there exists no so that $k_n = k_{n_0}$ for $n > n_0$. So $\dfrac{\log n}{\pi} = k_{n_0} + \lambda_n$ for $n > n_0$. Since $\lambda_n \to 0$ we get contradiction with $\log n \to \infty$.

Problem 4.

a) If we denote by E_{ij} the standard basis of M_n consisting of elementary matrix (with entry 1 at the place (i,j) and zero elsewhere), then the entries c_{ij} of C can be defined by $c_{ij} = f(E_{ji})$.

b) Denote by L the n^2-1-dimensional linear subspace of M_n consisting of allmatrices with zero trace. The elements E_{ij} with $i \neq j$ and the elements $E_{ii} - E_{nn}, i = 1, \ldots, n-1$ form a linear basis for L. Since

$$E_{ij} = E_{ij}.E_{jj} - E_{jj}E_{ij},\ i \neq j$$
$$E_{ii} - E_{nn} = E_{in}E_{ni} - E_{ni}E_{in},\ i = 1, \ldots, n-1,$$

then the property (2) shows that f is vanishing identically on L. Now, for any $A \in M_n$ we have $A - \frac{1}{n}tr(A).E \in L$, where E is the identity matrix, and therefore $f(A) = \frac{1}{n}f(E).tr(A)$.

Problem 5.

Let $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}, f^0 = id, f^{-n} = (f^{-1})^n$ for every natural number n. Let $T(x) = \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$. The sets $T(x)$ for different $x's$ either coinside or do not intersect. Each of them is mapped by f onto itself. It is enough to prove the theorem for every such set. Let $A = T(x)$. If A is finite, then we can think that A is the set of all vertices of a regular n polygon and that f is rotation by $\frac{2\pi}{n}$. Such rotation can be obtained as a composition of 2 symmetries mapping the n polygon onto itself (if n is even then there are axes of symmetry making $\frac{\pi}{n}$ angle; if $n = 2k+1$ then there are axes making $k\frac{2\pi}{n}$ angle). If A is infinite then we can think that $A = \mathbb{Z}$ and $f(m) = m + 1$ for every $m \in \mathbb{Z}$. In this case we define g_1 as a symmetry relative to $\frac{1}{2}$, g_2 as a symmetry relative to 0.

Problem 6.

a) $f(x) = x \sin \frac{1}{x}$.

b) Yes. The Cantor set is given by

$$C = \{x \in [0,1) : x = \sum_{j=1}^{\infty} b_j 3^{-j},\ b_j \in \{0,2\}\}.$$

There is an one-to-one mapping $f : [0,1) \to C$. Indeed, for $x = \sum\limits_{j=1}^{\infty} a_j 2^{-j},\ a_j \in \{0,1\}$ we set $f(x) = \sum\limits_{j=1}^{\infty} (2a_j)3^{-j}$. Hence C is uncountable.

For $k = 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, 2^{k-1} - 1$ we set

$$a_{k,i} = 3^{-k}\left(6\sum_{j=0}^{k-2} a_j 3^j + 1\right),\ b_{k,i} = 3^{-k}\left(6\sum_{j=0}^{k-2} a_j 3^j + 2\right),$$

where $i = \sum\limits_{j=0}^{k-2} a_j 2^j,\ a_j \in \{0,1\}$. Then

$$[0,1]\backslash C = \cup_{k=1}^{\infty} \cup_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i}),$$

i.e. the Cantor set consists of all points which have a trinary representation with 0 and 2 as digits and the points of its compliment have some $1's$ in their trinary representation. Thus, $\bigcup\limits_{i=0}^{2^{k-1}-1}(a_{k,i}, b_{k,i})$ are all points (exept $a_{k,i}$) which have 1 on k-th place and 0 or 2 on the j-th ($j < k$) places.

Noticing that the points with at least one digit equals to 1 are everywhere dence in $[0,1]$ we set

$$f(x) = \sum_{k=1}^{\infty}(-1)^k g_k(x).$$

where g_k is a piece-wise linear continuous functions with values at the knots

$$g_k\Big(\frac{a_{k,i}+b_{k,i}}{2}\Big) = 2^{-k},$$

$$g_k(0) = g_k(1) = g_k(a_{k,i}) = g_k(b_{k,i}) = 0, i = 0,1,\dots,2^{k-1}-1.$$

Then f is continuous and f "crosses the axis" at every point of the Cantor set.

2.5 Solutions of Olympic 1998

2.5.1 Day 1

Problem 1. First choose a basis $\{v_1, v_2, v_3\}$ of U_1. It is possible to extend this basis with vectors v_4, v_5 and v_6 to get a basis of U_2. In the same way we can extend a basis of U_2 with vectors $v_7,\dots,v_{10}$ to get as basis of V.

Let $T \in \epsilon$ be an endomorphism which has U_1 and U_2 as invariant subspaces. Then its matrix, relative to the basis $\{v_1,\dots,v_{10}\}$ is of the form

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}$$

So $dim_{\mathbb{R}}\epsilon = 9 + 18 + 40 = 67.$

Problem 2.

Let S_n be the group of permutations of $\{1,2,\dots,n\}$.

1) When $n = 3$ the proposition is obvious: if $x = (12)$ we choose $y = (123)$; if $x = (123)$ we choose $y = (12)$.

2) $n = 4$. Let $x = (12)(34)$. Assume that there exists $y \in S_n$, such that $S_4 =< x, y >$. Denote by K the invariant subgroup

$$K = \{id, (12)(34), (13)(24), (14)(23)\}.$$

By the fact that x and y generate the whole group S_4, it follows that the factor group S_4/K contains only powers of $\overline{y} = yK$, i.e., S_4/K is cyclic. It is easy to see that this factor-group is not comutative (something more this group is not isomorphic to S_3).

3) $n = 5$

a) If $x = (12)$, then for y we can take $y = (12345)$.

b) If $x = (123)$, we set $y = (124)(35)$. Then $y^3xy^3 = (125)$ and $y^4 = (124)$. Therefore $(123), (124), (125) \in< x, y >$- the subgroup generated by x and y. From the fact that $(123), (124), (125)$ generate the alternating subgroup A_5, it follows that $A_5 \subset< x, y >$. Moreover y is an odd permutation, hence $< x, y >= S_5$.

c) If $x = (123)(45)$, then as in b) we see that for y we can take the element (124).

d) If $x = (1234)$, we set $y = (12345)$. Then $(yx)^3 = (24) \in< x, y >$ $, x^2(24) = (13) \in< x, y >$ and $y^2 = (13524) \in< x, y >$. By the fact $(13) \in< x, y >$ and $(13524) \in< x, y >$, it follows that $< x, y >= S_5$.

e) If $x = (12)(34)$, then for y we can take $y = (1354)$. Then $y^2x = (125), y^3x = (124)(53)$ and by c) $S_5 =< x, y >$.

f) If $x = (12345)$, then it is clear that for y we can take the element $y = (12)$.

Problem 3. a) Fix $x = x_0 \in (0, 1)$. If we denote $x_n = f_n(x_0), n = 1, 2, \dots$ it is easy to see that $x_1 \in (0, 1/2], x_1 \leq f(x_1) \leq 1/2$ and $x_n \leq f(x_n) \leq 1/2$ (by induction). Then $(x_n)_n$ is a bounded nondecreasing sequence and, since $x_{n+1} = 2x_n(1 - x_n)$, the limit $l = \lim_{n\to\infty} x_n$ satisfies $l = 2l(1 - l)$, which implies $l = 1/2$. Now the monotone convergence theorem implies that

$$\lim_{n\to\infty}\int\limits_0^1 f_n(x)dx = \frac{1}{2}.$$

b) We prove by induction that

$$f_n(x) = \frac{1}{2} - 2^{2^n-1}\Big(1-\frac{1}{2}\Big)^{2^n} \qquad (1)$$

holds for $n = 1, 2, \ldots$. For $n = 1$ this is true, since $f(x) = 2x(1-x) = \frac{1}{2} - 2(x-\frac{1}{2})^2$. If (1) holds for some $n = k$, then we have

$$\begin{aligned} f_{k+1}(x) = f_k(f(x)) &= \frac{1}{2} - 2^{2^k-1}(\frac{1}{2} - 2(x-\frac{1}{2})^2) - \frac{1}{2})^{2^k} \\ &= \frac{1}{2} - 2^{2^k-1}(-2(x-\frac{1}{2})^2)^{2^k} \\ &= \frac{1}{2} - 2^{2^{k+1}-1}(x-\frac{1}{2})^{2^{k+1}} \end{aligned}$$

which is (2) for $n = k+1$.

Using (1) we can compute the integral,

$$\int\limits_0^1 f_n(x)dx = \Big[\frac{1}{2}x - \frac{2^{2^n-1}}{2^n+1}\Big(x-\frac{1}{2}\Big)^{2^n+1}\Big]_{x=0}^1 = \frac{1}{2} - \frac{1}{2(2^n+1)}.$$

Problem 4. Define the function

$$g(x) = \frac{1}{2}f^2(x) + f'(x).$$

Because $g(0) = 0$ and

$$f(x).f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number $0 < \eta \le 1$ for which $g(\eta) = 0$.

a) If f is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because $h(0) = h(1) = -\dfrac{1}{2}$, there exists a real number $0 < \eta < 1$ for which $h'(\eta) = 0$. But $g = f^2.h'$, and we are done.

b) If f has at least one zero, let z_1 be the first one and z_2 be the last one. (The set of the zeros is closed.) By the conditions, $0 < z_1 \le z_2 < 1$.

The function f is positive on the intervals $[0, z_1)$ and $(z_2, 1]$; this implies that $f'(z_1) \le 0$ and $f'(z_2) \ge 0$. Then $g(z_1) = f'(z_1) \le 0$ and $g(z_2) = f'(z_2) \ge 0$, and there exists a real number $\eta \in [z_1, z_2]$ for which $g(\eta) = 0$.

Remark. For the function $f(x) = \dfrac{2}{x+1}$ the conditions hold and $f.f' + f$" is constantly 0.

Problem 5. Observe that both sides of (2) are identically equal to zero if $n = 1$. Suppose that $n > 1$. Let $x_1, \dots, x_n$ be the zeros of P. Clearly (2) is true when $x = x_i$, $i \in \{1, \dots, n\}$, and equality is possible only if $P'(x_i) = 0$, i.e., if x_i is a multiple zero of P. Now suppose that x is not a zero of P. Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}, \quad \frac{P''(x)}{P(x)} = \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)},$$

we find

$$(n-1)\left(\frac{P'(x)}{P(x)}\right)^2 - n\frac{P''(x)}{P(x)} = \sum_{i=1}^{n} \frac{n-1}{(x - x_i)^2} - \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)}.$$

But this last expression is simply

$$\sum_{1 \le i < j \le n} \left(\frac{1}{x - x_i} - \frac{1}{x - x_j}\right)^2,$$

and therefore is positive. The inequality is proved. In order that (2) holds with equality sign for every real x it is necessary that $x_1 = x_2 = \ldots = x_n$. A direct verification shows that indeed, if $P(x) = c(x - x_1)^n$, then (2) becomes an identity.

Problem 6. Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta d\theta$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}.$$

Now let $f(x) = \sqrt{1-x^2}$. If $x = \sin\theta$ and $y = \sin\phi$ then

$$xf(y) + yf(x) = \sin\theta\cos\phi + \sin\phi\cos\theta = \sin(\theta+\phi) \le 1.$$

2.5.2 Day 2

Problem 1. We use induction on k. By passing to a subset, we may assume that $f_1, \dots, f_k$ are linearly independent.

Since f_k is independent of $f_1, \dots, f_{k-1}$, by induction there exists a vector $a_k \in V$ such that $f_1(a_k) = \cdots = f_{k-l}(a_k) = 0$ and $f_k(a_k) =\neq 0$. After normalising, we may assume that $f_k(a_k) = 1$. The vectors $a_1, \dots, a_{k-1}$ are defined similarly to get

$$f_i(a_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

For an arbitrary $x \in V$ and $1 \le i \le k$, $f_i(x - f_1(x)a_1 - f_2(x)a_2 - \cdots - f_k(x)a_k) = f_i(x) - \sum\limits_{j=1}^{k} f_j(x)f_i(a_j) = f_i(x) - f_i(x)f_i(a_i) = 0$, thus $f(x - f_1(x)a_1 - \cdots - f_k(x)a_k) = 0$. By the linearity of f this implies $f(x) = f_1(x)f(a_1) + \cdots + f_k(x)f(a_k)$, which gives $f(x)$ as a linear combination of $f_1(x), \dots, f_k(x)$.

Problem 2. Denote $x_0 = 1, x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 1,$

$$\omega(x) = \prod_{i=0}^{3}(x - x_i),$$
$$\omega_k(x) = \frac{\omega(x)}{x - x_k},\ k = 0, \ldots, 3,$$
$$l_k(x) = \frac{\omega_k(x)}{\omega_k(x_k)}.$$

Then for every $f \in \mathcal{P}$

$$f''(x) = \sum_{k=0}^{3} l_k''(x) f(x_k),$$
$$|f''(x)| \le \sum_{k=0}^{3} |l_k''(x)|.$$

Since f" is a linear function $\max_{-1\le x\le 1}|f''(x)|$ is attained either at $x = -1$ or at $x = 1$. Without loss of generality let the maximum point is $x = 1$. Then

$$\sup_{f\in\mathcal{P}} \max_{-1\le x\le 1} |f''(x)| = \sum_{k=0}^{3} |l_k''(1)|.$$

In order to have equality for the extremal polynomial f_* there must hold

$$f_*(x_k) = sign l_k''(1),\ k = 0, 1, 2, 3.$$

It is easy to see that $\{l_k''(1)\}_{k=0}^{3}$ alternate in sign, so $f_*(x_k) = (-1)^{k-1}, k = 0, \ldots, 3$. Hence $f_*(x_k) = T_3(x) = 4x^3 - 3x$, the Chebyshev polynomial of the first kind, and $f_*''(1) = 24$. The other extremal polynomial, corresponding to $x = -1$, is $-T_3$.

Problem 3. Let $f_n(x) = \underbrace{f(f(\ldots f}_{n}(p)))$. It is easy to see that $f_n(x)$ is a picewise monotone function and its graph contains 2^n linear segments; one endpoint is always on $\{(x, y) : 0 \le x \le 1, y = 0\}$, the other is on $\{(x, y) : 0 \le x \le 1, y = 1\}$. Thus the graph of the identity function intersects each segment once, so the number of points for which $f_n(x) = x$ is 2^n.

Since for each n-periodic points we have $f_n(x) = x$, the number of n-periodic points is finite.

A point x is n-periodic if $f_n(x) = x$ but $f_k(x) \neq x$ for $k = 1, \ldots, n-1$. But as we saw before $f_k(x) = x$ holds only at 2^k points, so there are at most $2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 2$ points x for which $f_k(x) = x$ for at least one $k \in \{1, 2, \ldots, n-1\}$. Therefore at least two of the 2^n points for which $f_n(x) = x$ are n-periodic points.

Problem 4. It is clear that $id : A_n \to A_n$ given by $id(x) = x$, does not verify condition (2). Since id is the only increasing injection on $A_n, \mathcal{F}$ does not contain injections. Let us take any $f \in \mathcal{F}$ and suppose that $\#(f^{-1}(k)) \geq 2$. Since f is increasing, there exists $i \in A_n$ such that $f(i) = f(i+1) = k$. In view of (2), $f(k) = f(f(i+1)) = f(i) = k$. If $\{i < k : f(i) < k\} = \emptyset$, then taking $j = \max\{i < k : f(i) < k\}$ we get $f(j) < f(j+1) = k = f(f(j+1))$, a contradiction. Hence $f(i) = k$ for $i \leq k$. If $\#(f^{-1}(\{l\})) \geq 2$ for some $l \geq k$, then the similar consideration shows that $f(i) = l = k$ for $i \leq k$. Hence $\#(f^{-1}\{i\}) = 0$ or 1 for every $i > k$. Therefore $f(i) \leq i$ for $i > k$. If $f(l) = l$, then taking $j = \max\{i < l : f(i) < l\}$ we get $f(j) < f(j+1) = l = f(f(j+1))$, a contradiction. Thus, $f(i) \leq i-1$ for $i > k$. Let $m = \max\{i : f(i) = k\}$. Since f is non-constant $m \leq n-1$. Since $k = f(m) = f(f(m+1)), f(m+1) \in [k+1, m]$. If $f(l) > l-1$ for some $l > m+1$, then $l-1$ and $f(l)$ belong to $f^{-1}(f(l))$ and this contradicts the facts above. Hence $f(i) = i-1$ for $i > m+1$. Thus we show that every function f in $\mathcal{F}$ is defined by natural numbers k, l, m, where $1 \leq k < l = f(m+1) \leq m \leq n-1$.

$$f(i) = \begin{cases} k & \text{if } i \leq m \\ l & \text{if } i = m \\ i-1 & \text{if } i > m+1 \end{cases}$$

Then

$$\#(\mathcal{F}) = \binom{n}{3}.$$

Problem 5. For every $x \in M$ choose spheres $S, T \in \mathcal{S}$ such that $S \neq T$ and $x \in S \cap T$; denote by U, V, W the three components of $\mathbb{R}^n \backslash (S \cup T)$, where the notation is such that $\partial U = S, \partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, and choose points with rational coordinates $u \in U, v \in V$, and $w \in W$. We claim that x is uniquely determined by the triple $< u, v, w >$; since the set of such triples is countable, this will finish the proof.

To prove the claim, suppose, that from some $x' \in M$ we arrived to the same $< u, v, w >$ using spheres $S', T' \in \mathcal{S}$ and components U', V', W' of $\mathbb{R}^n \backslash (S' \cup T')$. Since $S \cap S'$ contains at most one point and since $U \cap U' \neq \emptyset$, we have that $U \subset U'$ or $U' \subset U$; similarly for $V's$ and $W's$. Exchanging the role of x and x' and/or of $U's$ and $V's$ if necessary, there are only two cases to consider: (a) $U \supset U'$ and $V \supset V'$ and (b) $U \subset U', V \supset V'$ and $W \supset W'$. In case (a) we recall that $\overline{U} \cap \overline{V}$ contains only x and that $x' \in \overline{U'} \cap \overline{V'}$, so $x = x'$. In case (b) we get from $W \subset W'$ that $U' \subset \overline{U \cup V}$; so since U' is open and connected, and $\overline{U} \cap \overline{V}$ is just one point, we infer that $U' = U$ and we are back in the already proved case (a).

Problem 6.

a) We first construct a sequence c_n of positive numbers such that $c_n \to \infty$ and $\sum\limits_{n=1}^{\infty} c_n b_n < \frac{1}{2}$. Let $B = \sum\limits_{n=1}^{\infty} b_n$, and for each $k = 0, 1, \ldots$ denote by N_k the first positive integer for which

$$\sum_{n=N_k}^{\infty} b_n \leq \frac{B}{4^k}.$$

Now set $c_n = \dfrac{2^k}{5B}$ for each $n, N_k \leq n < N_{k+1}$. Then we have $c_n \to \infty$ and

$$\sum_{n=1}^{\infty} c_n b_n = \sum_{k=0}^{\infty} \sum_{N_k \leq n < N_{k+1}} c_n b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} \sum_{n=N_k}^{\infty} b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} . \frac{B}{4^k} = \frac{2}{5}.$$

Consider the intervals $I_n = (a_n - c_n b_n, a_n + c_n b_n)$. The sum of their lengths is $2 \sum c_n b_n < 1$, thus there exists a point $x_0 \in (0, 1)$ which is not contained in any I_n. We show that f is differentiable at x_0, and $f'(x_0) = 0$. Since x_0 is outside of the intervals $I_n, x_0 \neq a_n$ an for any n and $f(x_0) = 0$. For arbitrary $x \in (0,1)\backslash\{x_0\}$, if $x = a_n$ for some n, then

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| = \frac{f(a_n)-0}{|a_n-x_0|} \leq \frac{b_n}{c_n b_n} = \frac{1}{c_n},$$

otherwise $\dfrac{f(x)-f(x_0)}{x-x_0} = 0$. Since $c_n \to \infty$, this implies that for arbitrary $\epsilon > 0$ there are only finitely many $x \in (0,1)\backslash\{x_0\}$ for which

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < \epsilon$$

does not hold, and we are done.

Remark. The variation of f is finite, which implies that f is differentiable almost everywhere.

b) We remove the zero elements from sequence b_n. Since $f(x) = 0$ except for a countable subset of $(0,1)$, if f is differentiable at some point x_0, then $f(x_0)$ and $f'(x_0)$ must be 0.

It is easy to construct a sequence β_n satisfying $0 < \beta_n \leq b_n,\ b_n \to 0$ and $\sum\limits_{n=1}^{\infty} \beta_n = \infty$.

Choose the numbers $a_1, a_2, \ldots$ such that the intervals $I_n = (a_n - \beta_n, a_n + \beta_n)(n = 1, 2, \ldots)$ cover each point of $(0,1)$ infinitely many times (it is possible since the sum of lengths is $2\sum b_n = \infty$. Then for arbitrary $x_0 \in (0,1), f(x_0) = 0$ and $\epsilon > 0$ there is an n for which $\beta_n < \epsilon$ and $c_0 \in I_n$ which implies

$$\frac{|f(a_n)-f(x_0)|}{|a_n-x_0|} > \frac{b_n}{\beta_n} \geq 1.$$

2.6 Solutions of Olympic 1999

2.6.1 Day 1

Problem 1.

a) The diagonal matrix

$$A = \lambda I = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

is a solution for equation $A^3 = A + I$ if and only if $\lambda^3 = \lambda + 1$, because $A^3 - A - I = (\lambda^3 - \lambda - 1)I$. This equation, being cubic, has real solution.

b) It is easy to check that the polynomial $p(x) = x^3 - x - 1$ has a positive real root λ_1 (because $p(0) < 0$) and two conjugated complex roots λ_2 and λ_3 (one can check the discriminant of the polynomial, which is $\left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{2}\right)^2 = \frac{23}{108} > 0$, or the local minimum and maximum of the polynomial).

If a matrix A satisfies equation $A^3 = A + I$, then its eigenvalues can be only λ_1, λ_2 and λ_3. The multiplicity of λ_2 and λ_3 must be the same, because A is a real matrix and its characteristic polynomial has only real coefficients. Denoting the multiplicity of λ_1 by α and the common multiplicity of λ_2 and λ_3 by β,

$$det A = \lambda_1^\alpha \lambda_2^\beta \lambda_3^\beta = \lambda_1^\alpha . (\lambda_2 \lambda_3)^\beta.$$

Because λ_1 and $\lambda_2 \lambda_3 = |\lambda_2|^2$ are positive, the product on the right side has only positive factors.

Problem 2. No. For, let π be a permutation of $\mathbf{N}$ and let $N \in \mathbf{N}$. We shall argue that

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} > \frac{1}{9}.$$

In fact, of the $2N$ numbers $\pi(N+1), \ldots, \pi(3N)$ only N can be $\leq N$ so that at least N of them are $> N$. Hence

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} \geq \frac{1}{(3N)^2} \sum_{n=N+1}^{3N} \pi(n) > \frac{1}{9N^2} \cdot N.N = \frac{1}{9}.$$

Solution 2. Let π be a permutation of $\mathbf{N}$. For any $n \in \mathbf{N}$, the numbers $\pi(1), \ldots, \pi(n)$ are distinct positive integers, thus $\pi(1) + \cdots +$

$\pi(n) \geq 1 + \cdots + n = \dfrac{n(n+1)}{2}$. By this inequality,

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} = \sum_{n=1}^{\infty} (\pi(1) + \cdots + p(n)) \Big(\frac{1}{n^2} - \frac{1}{(n+1)^2} \Big) \geq$$
$$\geq \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \cdot \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

Problem 3. Writing (1) with $n-1$ instead of n,

$$\Big| \sum_{k=1}^{n-1} 3^k (f(x+ky) - f(x-ky)) \Big| \leq 1. \tag{2}$$

From the difference of (1) and (2),

$$|3^n (f(x+ny) - f(x-ny))| \leq 2;$$

which means

$$|f(x+ny) - f(x-ny)| \leq \frac{2}{3^n}. \tag{3}$$

For arbitrary $u, v \in \mathbf{R}$ and $n \in \mathbf{N}$ one can choose x and y such that $x - ny = u$ and $x + ny = v$, namely $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2n}$, Thus, (3) yields

$$|f(u) - f(v)| \leq \frac{2}{3^n}$$

for arbitrary positive integer n. Because $\dfrac{2}{3^n}$ can be arbitrary small, this implies $f(u) = f(v)$.

Problem 4. Let $g(x) = \dfrac{f(x)}{x}$. We have $f\Big(\dfrac{x}{g(x)}\Big) = g(x)$. By induction it follows that $g\Big(\dfrac{x}{g^n(x)}\Big) = g(x)$, i.e.

$$f\Big(\frac{x}{g^n(x)}\Big) = \frac{x}{g^{n-1}(x)}, \; n \in \mathbf{N}. \tag{1}$$

On the other hand, let substitute x by $f(x)$ in $f\Big(\dfrac{x^2}{f(x)}\Big) = x$. From the injectivity of f we get $\dfrac{f^2(x)}{f(f(x))} = x$, i.e. $g(xg(x)) = g(x)$. Again by induction we deduce that $g(xg^n(x)) = g(x)$ which can be written in the form

$$f(xg^n(x)) = xg^{n-1}(x), n \in \mathbf{N}. \tag{2}$$

Set $f^{(m)} = \underbrace{f \circ f \circ \cdots \circ f}_{m \text{ times}}$. It follows from (1) and (2) that

$$f^{(m)}(xg^n(x)) = xg^{n-m}(x), m, n \in \mathbf{N}. \tag{3}$$

Now, we shall prove that g is a constant. Assume $g(x_1) < g(x_2)$. Then we may find $n \in \mathbf{N}$ such that $x_1g^n(x_1) \leq x_2g^n(x_2)$. On the other hand, if m is even then $f^{(m)}$ is strictly increasing and from (3) it follows that $x_1^m g^{n-m}(x_1) \leq x_2^m g^{n-m}(x_2)$. But when n is fixed the opposite inequality holds $\forall m \gg 1$. This contradiction shows that g is a constant, i.e. $f(x) = Cx, C > 0$.

Conversely, it is easy to check that the functions of this type verify the conditions of the problem.

Problem 5.

We prove the more general statement that if at least $n+k$ points are marked in an $n \times k$ grid, then the required sequence of marked points can be selected.

If a row or a column contains at most one marked point, delete it. This decreases $n+k$ by 1 and the number of the marked points by at most 1, so the condition remains true. Repeat this step until each row and column contains at least two marked points. Note that the condition implies that there are at least two marked points, so the whole set of marked points cannot be deleted.

We define a sequence $b_1, b_2, \ldots$ of marked points. Let b_1 be an arbitrary marked point. For any positive integer n, let b_{2n} be an other marked point in the row of b_{2n-1} and b_{2n+1} be an other marked point in the column of b_{2n}.

Let m be the first index for which b_m is the same as one of the earlier points, say $b_m = b_l$, $l < m$.

If $m-l$ is even, the line segments $b_l b_{l+1}, b_{l+1}b_{l+2}, \dots, b_{m-1}b_l = b_{m-1}b_m$ are alternating horizontal and vertical. So one can choose $2k = m-l$, and $(a_1, \dots, a_{2k}) = (b_l, \dots, b_{m-1})$ or $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_m)$ if l is odd or even, respectively.

If $m-l$ is odd, then the points $b_1 = b_m, b_{l+1}$ and b_{m-1} are in the same row/column. In this case chose $2k = m-l-1$. Again, the line segments $b_{l+1}b_{l+2}, b_{l+2}b_{l+3}, \dots, b_{m-1}b_{l+1}$ are alternating horizontal and vertical and one can choose $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_{m-1})$ or $(a_1, \dots, a_{2k}) = b_{l+2}, \dots, b_{m-1}, b_{l+1}$ if l is even or odd, respectively.

Solution 2. Define the graph G in the following way: Let the vertices of G be the rows and the columns of the grid. Connect a row r and a column c with an edge if the intersection point of r and c is marked.

The graph G has $2n$ vertices and $2n$ edges. As is well known, if a graph of N vertices contains no circle, it can have at most $N-1$ edges. Thus G does contain a circle. A circle is an alternating sequence of rows and columns, and the intersection of each neighbouring row and column is a marked point. The required sequence consists of these intersection points.

Problem 6.

a) Let $g(x) = \max(0, f'(x))$. Then $0 < \int\limits_{-1}^{1} f'(x)dx = \int\limits_{-1}^{1} g(x)dx + \int\limits_{-1}^{1} (f'(x)-g(x))dx$, so we get $\int\limits_{-1}^{1} |f'(x)|dx = \int\limits_{-1}^{1} g(x)dx + \int\limits_{-1}^{1} (g(x)-f'(x))dx < 2\int\limits_{-1}^{1} g(x)dx$. Fix p and c (to be determined at the end). Given any $t > 0$, choose for every x such that $g(x) > t$ an interval $I_x = [x, y]$ such that $|f(y) - f(x)| > cg(x)^{1/p}|y-x| > ct^{1/p}|I_x|$ and choose disjoint I_{x_i}. that cover at least one third of the measure of the set $\{g > t\}$. For $I = \bigcup\limits_i I_i$ we thus have $ct^{1/p}|I| \le \int\limits_I f'(x)dx \le \int\limits_{-1}^{1} |f'(x)|dx < 2\int\limits_{-1}^{1} g(x)dx$; so $|\{g > t\}| \le 3|I| < \frac{6}{c}t^{-1/p}\int\limits_{-1}^{1} g(x)dx$. Integrating the inequality, we get $\int\limits_{-1}^{1} g(x)dx = \int\limits_{0}^{1} |\{g > t\}|dt < \frac{6}{c}\frac{p}{p-1}\int\limits_{-1}^{1} g(x)dx$; this is a contradiction e.g. for $c_p = (6p)/(p-1)$.

b) No. Given $c > 1$, denote $\alpha = \frac{1}{c}$ and choose $0 < \epsilon < 1$ such that $\left(\frac{1+\epsilon}{2\epsilon}\right)^{-\alpha} < \frac{1}{4}$. Let $g : [-1,1] \to [-1,1]$ be continuous, even, $g(x) = -1$ for $|x| \le \epsilon$ and $0 \le g(x) < \alpha\left(\frac{|x|+\epsilon}{2\epsilon}\right)^{-\alpha-1}$ for $\epsilon < |x| \le 1$ is chosen such that $\int\limits_{\epsilon}^{1} g(t)dt > -\frac{\epsilon}{2} + \int\limits_{\epsilon}^{1} \alpha\left(\frac{|x|+\epsilon}{2\epsilon}\right)^{-\alpha-1} dt = -\frac{\epsilon}{2} + 2\epsilon\left(1 - \left(\frac{1+\epsilon}{2\epsilon}\right)^{-\alpha}\right) > \epsilon$. Let $f = \int g(t)dt$. Then $f(1) - f(-1) \ge -2\epsilon + 2\int\limits_{\epsilon}^{1} g(t)dt > 0$. If $\epsilon < x < 1$ and $y = -\epsilon$, then $|f(x) - f(y)| \ge 2\epsilon - \int\limits_{\epsilon}^{x} g(t)dt \ge 2\epsilon - \int\limits_{\epsilon}^{x} \alpha\left(\frac{t+\epsilon}{2\epsilon}\right)^{-\alpha-1} = 2\epsilon\left(\frac{x+\epsilon}{2\epsilon}\right)^{-\alpha} > g(x)\frac{|x-y|}{\alpha} = f'(x)\frac{|x-y|}{\alpha}$; symmetrically for $-1 < x < -\epsilon$ and $y = \epsilon$.

2.6.2 Day 2

Problem 1. From $0 = (a+b)^2 = a^2 + b^2 + ab + ba = ab + ba$, we have $ab = -(ba)$ for arbitrary a, b, which implies

$$abc = a(bc) = -((bc))a = -(b(ca)) = (ca)b = c(ab) = -((ab)c) = -abc.$$

Problem 2. For all nonnegative integers n and modulo 5 residue class r, denote by $p_n^{(r)}$ the probability that after n throwing the sum of values is congruent to r modulo n. It is obvious that $p_0^{(0)} = 1$ and $p_0^{(1)} = p_0^{(2)} = p_0^{(3)} = p_0^{(4)} = 0$.

Moreover, for any $n > 0$ we have

$$p_n^{(r)} = \sum_{i=1}^{6} \frac{1}{6} p_{n-1}^{(r-i)}. \tag{1}$$

From this recursion we can compute the probabilities for small values of n and can conjecture that $p_n^{(r)} = \frac{1}{5} + \frac{4}{5.6^n}$ if $n \equiv r \pmod 5$ and $p_n^{(r)} = \frac{1}{5} - \frac{1}{5.6^n}$ otherwise. From (1), this conjecture can be proved by induction.

Solution 2. Let S be the set of all sequences consisting of digits $1, \dots, 6$ of length n. We create collections of these sequences.

Let a collection contain sequences of the form

$$\underbrace{66\dots6}_{k} XY_1 \dots Y_{n-k-1},$$

where $X \in \{1, 2, 3, 4, 5\}$ and k and the digits $Y_1, \dots, Y_{n-k-1}$ are fixed. Then each collection consists of 5 sequences, and the sums of the digits of sequences give a whole residue system mod 5.

Except for the sequence $66\dots6$, each sequence is the element of one collection. This means that the number of the sequences, which have a sum of digits divisible by 5, is $\frac{1}{5}(6^n - 1) + 1$ if n is divisible by 5, otherwise $\frac{1}{5}(6^n - 1)$.

Thus, the probability is $\frac{1}{5} + \frac{4}{5.6^n}$ if n is divisible by 5, otherwise it is $\frac{1}{5} - \frac{1}{5.6^n}$.

Solution 3. For arbitrary positive integer k denote by p_k the probability that the sum of values is k. Define the generating function

$$f(x) = \sum_{k=1}^{\infty} p_k x^k = \left(\frac{x + x^2 + x^3 + x^4 + x^5 + x^6}{6}\right)^n.$$

(The last equality can be easily proved by induction.)

Our goal is to compute the sum $\sum\limits_{k=1}^{\infty} p_{5k}$. Let $\epsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$; be the first 5th root of unity. Then

$$\sum_{k=1}^{\infty} p_{5k} = \frac{f(1) + f(\epsilon) + f(\epsilon^2) + f(\epsilon^3) + f(\epsilon^4)}{5}.$$

Obviously $f(1) = 1$, and $f(\epsilon^j) = \frac{\epsilon^{jn}}{6^n}$ for $j = 1, 2, 3, 4$. This implies that

$f(\epsilon)+f(\epsilon^2)+f(\epsilon^3)+f(\epsilon^4)$ is $\dfrac{4}{6^n}$ if n is divisible by 5, otherwise it is $\dfrac{-1}{6^n}$.
Thus, $\sum_{k=1}^{\infty} p_{5k}$ is $\dfrac{1}{5}+\dfrac{1}{5.6^n}$ if n is divisible by 5, otherwise it is $\dfrac{1}{5}-\dfrac{1}{5.6^n}$.

Problem 3.

The inequality

$$0 \le x^3 - \frac{3}{4}x + \frac{1}{4} = (x+1)\Big(x-\frac{1}{2}\Big)^2$$

holds for $x \ge -1$.

Substituting $x_1, \dots, x_n$, we obtain

$$0 \le \sum_{i=1}^{n}\Big(x_i^3 - \frac{3}{4}x_i + \frac{1}{4}\Big) = \sum_{i=1}^{n} x_i^3 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4} = 0 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4},$$

so $\sum_{i=1}^{n} x_i \le \dfrac{n}{3}$. *Remark.* Equailty holds only in the case when $n = 9k, k$ of the $x_1, \dots, x_n$ are -1, and $8k$ of them are $\dfrac{1}{2}$.

Problem 4. Assume that such a function exists. The initial inequality can be written in the form $f(x)-f(x+y) \ge f(x) - \dfrac{f^2(x)}{f(x)+y} = \dfrac{f(x)y}{f(x)+y}$. Obviously, f is a decreasing function. Fix $x > 0$ and choose $n \in \mathbf{N}$ such that $nf(x+1) \ge 1$. For $k = 0, 1, \dots, n-1$ we have

$$f\Big(x+\frac{k}{n}\Big) - f\Big(x+\frac{k+1}{n}\Big) \ge \frac{f(x+\frac{k}{n})}{nf(x+\frac{k}{n})+1} \ge \frac{1}{2n}.$$

The additon of these inequalities gives $f(x+1) \ge f(x) - \dfrac{1}{2}$. From this it follows that $f(x+2m) \le f(x) - m$ for all $m \in \mathbf{N}$. Taking $m \ge f(x)$, we get a contradiction with the conditon $f(x) > 0$.

Problem 5. Let S be the set of all words consisting of the letters x, y, z, and consider an equivalence relation $\sim$ on S satisfying the following conditions: for arbitrary words $u, v, w \in S$

(i) $uu \sim u$;

(ii) if $v \sim w$, then $uv \sim uw$ and $vu \sim wu$.

Show that every word in S is equivalent to a word of length at most 8. (20 points)

Solution. First we prove the following lemma: If a word $u \in S$ contains at least one of each letter, and $v \in S$ is an arbitrary word, then there exists a word $w \in S$ such that $uvw \sim u$.

If v contains a single letter, say x, write u in the form $u = u_1xu_2$, and choose $w = u_2$. Then $uvw = u_1xu_2)xu_2 = u_1((xu_2)(xu_2)) \sim u_1(xu_2) = u$.

In the general case, let the letters of v be $a_1, \ldots, a_k$. Then one can choose some words $w_1, \ldots, w_k$ such that $(ua_1)w_1) \sim u, (ua_1a_2)w_2 \sim ua_1, \ldots, (ua_1 \ldots a_k)w_k \sim ua_1 \ldots a_{k-1}$. Then $u \sim ua_1w_1 \sim ua_1a_2w_2w_1 \sim \cdots \sim ua_1 \ldots a_kw_k \ldots w_1 = uv(w_k \ldots w_1)$, so $w = w_k \ldots w_1$ is a good choice.

Consider now an arbitrary word a, which contains more than 8 digits. We shall prove that there is a shorter word which is equivalent to a. If a can be written in the form $uvvw$, its length can be reduced by $uvvw \sim uvw$. So we can assume that a does not have this form.

Write a in the form $a = bcd$, where band d are the first and last four letter of a, respectively. We prove that $a \sim bd$.

It is easy to check that b and d contains all the three letters x, y and z, otherwise their length could be reduced. By the lemma there is a word e such that $b(cd)e \sim b$, and there is a word f such that $def \sim d$. Then we can write

$$a = bcd \sim bc(def) \sim bc(dedef) = (bcde)(def) \sim bd.$$

Remark. Of course, it is enough to give for every word of length 9 an shortest shorter word. Assuming that the first letter is x and the second is y, it is easy (but a little long) to check that there are 18 words of length 9 which cannot be written in the form $uvvw$.

For five of these words there is a 2-step solution, for example

$$xyxzyzx\underline{zy} \sim xy\underline{xzyzxzyzy} \sim xyx\underline{zyzy} \sim xyxzy.$$

In the remaining 13 cases we need more steps. The general algorithm given by the Solution works for these cases as well, but needs also very long words. For example, to reduce the length of the word $a = xyzyxzzxyz$, we have set $b = xyzy, c = x, d = zxyz, e = xyxzzxzyxyzy, f = zyxyxzyxzxzxzxyxyzxyz$. The longest word in the algorithm was

$$bcdedef =$$

$$= xyzyxzxyzxyxzxzyxyzyzxyzxyxzxzyxyzyzyxyxzyxzxzxzxyxyzxyz,$$

which is of length 46. This is not the shortest way: reducing the length of word a can be done for example by the following steps:

$$xyzyxzzx\underline{yz} \sim xyzyxzz\underline{xyzyx} \sim xyzyxzzxy\underline{zyx}yzyz \sim$$
$$\underline{xyzyxzzxyzyxz}yx\underline{yzyz} \sim\sim xy\underline{zyxzyx}yz \sim xyzyxyz.$$

(The last example is due to Nayden Kambouchev from Sofia University.)

Problem 6. Let $A = \{a_1, \ldots, a_k\}$. Consider the k-tuples

$$\left(exp\frac{2\pi i a_1 t}{n}, \ldots, exp\frac{2\pi i a_k t}{n}\right) \in \mathbf{C}^k,\ t = 0, 1, \ldots, n-1.$$

Each component is in the unit circle $|z| = 1$. Split the circle into 6 equal arcs. This induces a decomposition of the k-tuples into 6^k classes. By the condition $k \le \frac{1}{100}\ln$ we have $n > 6^k$, so there are two k-tuples in the same class say for $t_1 < t_2$. Set $r = t_2 - t_1$. Then

$$Re exp\frac{2\pi i a_j r}{n} = \cos\left(\frac{2\pi a_j t_2}{n} - \frac{2\pi a_j t_1}{n}\right) \ge \cos\frac{\pi}{3} = \frac{1}{2}$$

for all j, so

$$|f(r)| \ge Re f(r) \ge \frac{k}{2}.$$

2.7 Solutions of Olympic 2000

2.7.1 Day 1

Problem 1.

a) Yes.

Proof: Let $A = \{x \in [0,1] : f(x) > x\}$. If $f(0) = 0$ we are done, if not then A is non-empty (0 is in A) bounded, so it has supremum, say a. Let $b = f(a)$.

I. case: $a < b$. Then, using that f is monotone and a was the sup, we get $b = f(a) \leq f(\frac{(a+b)}{2}) \leq \frac{a+b}{2}$, which contradicts $a < b$.

II. case: $a > b$. Then we get $b = f(a) \geq f(\frac{a+b}{2}) > \frac{a+b}{2}$ contradiction. Therefore we must have $a = b$.

b) No. Let, for example,

$$f(x) = 1 - \frac{x}{2} \text{ if } x \leq \frac{1}{2}$$

and

$$f(x) = \frac{1}{2} - \frac{x}{2} \text{ if } x > \frac{1}{2}$$

This is clearly a good counter-example.

Problem 2. Short solution. Let

$$P(x, y) = \frac{p(x) - p(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 1$$

and

$$Q(x, y) = \frac{q(x) - q(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x + y.$$

We need those pairs (w, z) which satisfy $P(w, z) = Q(w, z) = 0$.

From $P - Q = 0$ we have $w + z = 1$. Let $c = wz$. After a short calculation we obtain $c^2 - 3c + 2 = 0$, which has the solutions $c = 1$ and $c = 2$. From the system $w + z = 1, wz = c$ we obtain the following pairs:

$$\Big(\frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2}\Big) \text{ and } \Big(\frac{1 \pm \sqrt{7}i}{2}, \frac{1 \mp \sqrt{7}i}{2}\Big).$$

Problem 3.

A and B are square complex matrices of the same size and

$$rank(AB - BA) = 1.$$

Show that $(AB - BA)^2 = 0$.

Let $0 = AB - BA$. Since $rankC = 1$, at most one eigenvalue of C is different from 0. Also $trC = 0$, so all the eigevalues are zero. In the Jordan canonical form there can only be one 2×2 cage and thus $C^2 = 0$.

Problem 4.

a)

$$\Big(\sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\Big)^2=\sum_{i,j}^{n}\frac{x_ix_j}{\sqrt{i}\sqrt{j}}\geq\sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\sum_{j=1}^{i}\frac{x_i}{\sqrt{j}}\geq\sum_{i=1}^{n}\frac{xi}{\sqrt{i}}i\frac{x_i}{\sqrt{i}}=\sum_{i=1}^{n}x_i^2$$

b)

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\Big(\sum_{i=m}^{\infty}x_i^2\Big)^{1/2}\leq\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{i=m}^{\infty}\frac{x_i}{\sqrt{i-m+1}}$$

by a)

$$=\sum_{i=1}^{\infty}x_i\sum_{m=1}^{i}\frac{1}{\sqrt{m}\sqrt{i-m+1}}$$

You can get a sharp bound on

$$\sup_i\sum_{m=1}^{i}\frac{1}{\sqrt{m}\sqrt{i-m+1}}$$

by checking that it is at most

$$\int\limits_0^{i+1}\frac{1}{\sqrt{x}\sqrt{i+1-x}}dx=\pi$$

Alternatively you can observe that

$$\sum_{m=1}^{i}\frac{1}{\sqrt{m}\sqrt{i+1-m}}=2\sum_{m=1}^{i/2}\frac{1}{\sqrt{m}\sqrt{i+1-m}}\leq$$

$$\leq 2\frac{1}{\sqrt{\frac{i}{2}}}\sum_{m=1}^{i/2}\frac{1}{\sqrt{m}}\leq 2\frac{1}{\sqrt{\frac{i}{2}}}.2\sqrt{\frac{i}{2}}=4$$

Problem 5.

Suppose that $e+f+g=0$ for given idempotents $e,f,g\in R$. Then

$$g=g^2=(-(e+f)^2=e+(ef+fe)+f=(ef+fe)-g,$$

i.e. $ef+fe=2g$, whence the additive commutator

$$[e,f]=ef-fe=[e,ef+fe]=2[e,g]=2[e,-e-f]=-2[e,f],$$

i.e. $ef = fe$ (since R has zero characteristic). Thus $ef + fe = 2g$ becomes $ef = g$, so that $e + f + ef = 0$. On multiplying by e, this yields $e + 2ef = 0$, and similarly $f + 2ef = 0$, so that $f = -2ef = e$, hence $e = f = g$ by symmetry. Hence, finaly, $3e = e + f + g = 0$, i.e. $e = f = g = 0$.

For part (i) just omit some of this.

Problem 6.

From the conditions it is obvious that F is increasing and $\lim_{n\to\infty} b_n = \infty$.

By Lagrange's theorem and the recursion in (1), for all $k \geq 0$ integers there exists a real number $\xi \in (a_k, a_{k+1})$ such that

$$F(a_{k+1}) - F(a_k) = f(\xi)(a_{k+1} - a_k) = \frac{f(\xi)}{f(a_k)}. \tag{2}$$

By the monotonity, $f(a_k) \leq f(\xi) \leq f(a_{k+1})$, thus

$$1 \leq F(a_{k+1}) - F(a_k) \leq \frac{f(a_{k+1})}{f(a_k)} = 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{3}$$

Summing (3) for $k = 0, \dots, n-1$ and substituting $F(b_n) = n$, we have

$$F(b_n) < n + F(a_0) \leq F(a_n) \leq F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{4}$$

From the first two inequalities we already have $a_n > b_n$ and $\lim_{n\to\infty} a_n = \infty$.

Let ϵ be an arbitrary positive number. Choose an integer K_ϵ such that $f(aK_\epsilon) > \frac{2}{\epsilon}$. If n is sufficiently large, then

$$F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} =$$

$$= \left(F(a_0) + \sum_{k=0}^{K_\epsilon - 1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}\right) + \sum_{k=K_\epsilon} n - 1 \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} < \tag{5}$$

$$< O_\epsilon(1) + \frac{1}{f(aK_\epsilon)} \sum_{k=K_\epsilon}^{n-1} (f(a_{k+1}) - f(a_k)) <$$

$$< O_\epsilon(1) + \frac{\epsilon}{2}(f(a_n) - f(aK_\epsilon)) < \epsilon f(a_n).$$

Inequalities (4) and (5) together say that for any positive ϵ, if n is sufficiently large,

$$F(a_n) - F(b_n) < \epsilon f(a_n).$$

Again, by Lagrange's theorem, there is a real number $\zeta \in (b_n, a_n)$ such that

$$F(a_n) - F(b_n) = f(\zeta)(a_n - b_n) > f(b_n)(a_n - b_n), \tag{6}$$

thus

$$f(b_n)(a_n - b_n) < \epsilon f(a_n). \tag{7}$$

Let B be an upper bound for f'. Apply $f(a_n) < f(b_n) + B(a_n - b_n)$ in (7):

$$f(b_n)(a_n - b_n) < \epsilon(f(b_n) + B(a_n - b_n)),$$
$$(f(b_n) - \epsilon B)(a_n - b_n) < \epsilon f(b_n). \tag{8}$$

Due to $\lim_{n\to\infty} f(b_n) = \infty$, the first factor is positive, and we have

$$a_n - b_n < \epsilon \frac{f(b_n)}{f(b_n) - \epsilon B} < 2\epsilon \tag{9}$$

for sufficiently large n.

Thus, for arbitrary positive ϵ we proved that $0 < a_n - b_n < 2\epsilon$ if n is sufficiently large.

2.7.2 Day 2

Problem 1.

We start with the following lemma: If a and b be coprime positive integers then every sufficiently large positive integer m can be expressed in the form $ax + by$ with x, y non-negative integers.

Proof of the lemma. The numbers $0, a, 2a, \ldots, (b-1)a$ give a complete residue system modulo b. Consequently, for any m there exists a $0 \le x \le b-1$ so that $ax \equiv m \pmod b$. If $m \ge (b-1)a$, then $y = (m-ax)/b$, for which $x + by = m$, is a non-negative integer, too.

Now observe that any dissection of a cube into n smaller cubes may be refined to give a dissection into $n+(a^d-1)$ cubes, for any $a \geq 1$. This refinement is achieved by picking an arbitrary cube in the dissection, and cutting it into a^d smaller cubes. To prove the required result, then, it suffices to exhibit two relatively prime integers of form $a^d - 1$. In the 2-dimensional case, $a_1 = 2$ and $a_2 = 3$ give the coprime numbers $2^2-1 = 3$ and $3^2 - 1 = 8$. In the general case, two such integers are $2^d - 1$ and $(2^d - l)^d - 1$, as is easy to check.

Problem 2. Let $(x-\alpha, x+\alpha) \subset [0,1]$ be an arbitrary non-empty open interval. The function f is not monoton in the intervals $[x-\alpha, x]$ and $[x, x+\alpha]$, thus there exist some real numbers $x-\alpha \leq p < q \leq x, x \leq r < s \leq x+\alpha$ so that $f(p) > f(q)$ and $f(r) < f(s)$.

By Weierstrass' theorem, f has a global minimum in the interval $[p, s]$. The values $f(p)$ and $f(s)$ are not the minimum, because they are greater than $f(q)$ and $f(s)$, respectively. Thus the minimum is in the interior of the interval, it is a local minimum. So each nonempty interval $(x-\alpha, x+\alpha) \subset [0,1]$ contains at least one local minimum.

Problem 3. The statement is not true if p is a constant polynomial. We prove it only in the case if n is positive.

For an arbitrary polynomial $q(z)$ and complex number c, denote by $\mu(q, c)$ the largest exponent α for which $q(z)$ is divisible by $(z-c)^\alpha$. (With other words, if c is a root of q, then $\mu(q, c)$ is the root's multiplicity. Otherwise 0.)

Denote by S_0 and S_1 the sets of complex numbers z for which $p(z)$ is 0 or 1, respectively. These sets contain all roots of the polynomials $p(z)$ and $p(z) - 1$, thus

$$\sum_{c \in S_0} \mu(p, c) = \sum_{c \in S_1} \mu(p-1, c) = n. \tag{1}$$

The polynomial p' has at most $n-1$ roots ($n > 0$ is used here). This implies that

$$\sum_{c\in S_0\cup S_1} \mu(p',c) \le n-1. \tag{2}$$

If $p(c)=0$ or $p(c)-1=0$, then

$$\mu(p,c)-\mu(p'c)=1 \text{ or } \mu(p-1,c)-\mu(p'c)=1, \tag{3}$$

respectively. Putting (1), (2) and (3) together we obtain

$$|S_0|+|S_1| = \sum_{c\in S_0}(\mu(p,c)-\mu(p',c)) + \sum_{c\in S_1}(\mu(p-1,c)-\mu(p',c)) =$$
$$= \sum_{c\in S_0}\mu(p,c) + \sum_{c\in S_1}\mu(p-1,c) - \sum_{c\in S_0\cup S_1}\mu(p',c) \ge n+n-(n-1) = n+1.$$

Problem 4.

a) Without loss of generality, we can assume that the point A_2 is the origin of system of coordinates. Then the polynomial can be presented in the form

$$y = (a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4)x^2 + a_5x,$$

where the equation $y = a_5x$ determines the straight line A_1A_3. The abscissas of the points A_1 and A_3 are $-a$ and $a, a > 0$, respectively. Since $-a$ and a are points of tangency, the numbers $-a$ and a must be double roots of the polynomial $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$. It follows that the polynomial is of the form

$$y = a_0(x^2-a^2)^2 + a_5x.$$

The equality follows from the equality of the integrals

$$\int_{-a}^{0} a_0(x^2-a^2)x^2dx = \int_{0}^{a} a_0(x^2-a^2)x^2dx$$

due to the fact that the function $y = a_0(x^2-a^2)$ is even.

b) Without loss of generality, we can assume that $a_0 = 1$. Then the function is of the form

$$y = (x+a)^2(x-b)^2x^2 + a_5x,$$

where a and b are positive numbers and $b = ka, 0 < k < \infty$. The areas of the figures at the segments A_1A_2 and A_2A_3 are equal respectively to

$$\int_{-a}^{0} (x+a)^2(x-b)^2x^2dx = \frac{a^7}{210}(7k^2+7k+2)$$

and

$$\int_{0}^{b} (x+a)^2(x-b)^2x^2dx = \frac{a^7}{210}(2k^2+7k+7)$$

Then

$$K = k^5\frac{2k^2+7k+7}{7k^2+7k+2}.$$

The derivative of the function $f(k) = \dfrac{2k^2+7k+7}{7k^2+7k+2}$ is negative for $0 < k < \infty$. Therefore $f(k)$ decreases from $\dfrac{7}{2}$ to $\dfrac{2}{7}$ when k increases from 0 to ∞. Inequalities $\dfrac{2}{7} < \dfrac{2k^2+7k+7}{7k^2+7k+2} < \dfrac{7}{2}$ imply the desired inequalities.

Problem 5.

First solution. First, if we assume that $f(x) > 1$ for some $x\mathbb{R}^+$, setting $y = \dfrac{x}{f(x)-1}$ gives the contradiction $f(x) = 1$. Hence $f(x) \leq 1$ for each $x \in \mathbb{R}^+$, which implies that f is a decreasing function.

If $f(x) = 1$ for some $x \in \mathbb{R}^+$, then $f(x+y) = f(y)$ for each $y \in \mathbb{R}^+$, and by the monotonicity of f it follows that $f \equiv 1$.

Let now $f(x) < 1$ for each $x \in \mathbb{R}^+$. Then f is strictly decreasing function, in particular injective. By the equalities

$$f(x)f(yf(x)) = f(x+y) =$$
$$= f(yf(x)+x+y(1-f(x))) = f(yf(x))f((x+y(1-f(x)))f(yf(x)))$$

we obtain that $x = (x+y(1-f(x)))f(yf(x))$. Setting $x = 1, z = xf(1)$ and $a = \dfrac{1-f(1)}{f(1)}$, we get $f(z) = \dfrac{1}{1+az}$.

Combining the two cases, we conclude that $f(x) = \dfrac{1}{1+ax}$ for each

$x \in \mathbb{R}^+$, where $a \geq 0$. Conversely, a direct verification shows that the functions of this form satisfy the initial equality.

Second solution. As in the first solution we get that f is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x+y)-f(x)}{y} = f^2(x)\frac{f(yf(x))-1}{yf(x)}.$$

It follows that if f is differentiable at the point $x \in \mathbb{R}^+$, then there exists the limit $\lim\limits_{z\to 0^+}\dfrac{f(z)-1}{z} =: -a$. Therefore $f'(x) = -af^2(x)$ for each $x \in \mathbb{R}^+$, i.e. $\left(\dfrac{1}{f(x)}\right)' = a$, which means that $f(x) = \dfrac{1}{ax+b}$. Substituting in the initial relaton, we find that $b=1$ and $a \geq 0$.

Problem 6. First we prove that for any polynomial q and $m \times m$ matrices A and B, the characteristic polinomials of $q(e^{AB})$ and $q(e^{BA})$ are the same. It is easy to check that for any matrix X, $q(e^X) = \sum\limits_{n=0}^{\infty} c_n X^n$ with some real numbers c_n which depend on q. Let

$$C = \sum_{n=1}^{\infty} c_n.(BA)^{n-1}B = \sum_{n=1}^{\infty} c_n.B(AB)^{n-1}.$$

Then $q(e^{AB}) = c_0 I + AC$ and $q(e^{BA}) = c_0 I + CA$. It is well-known that the characteristic polynomials of AC and CA are the same; denote this polynomial by $f(x)$. Then the characteristic polynomials of matrices $q(e^{AB})$ and $q(e^{BA})$ are both $f(x - c_0)$.

Now assume that the matrix $p(e^{AB})$ is nilpotent, ie. $(p(e^{AB}))^k = 0$ for some positive integer k. Chose $q = p^k$. The characteristic polynomial of the matrix $q(^{AB}) = 0$ is x^m, so the same holds for the matrix $q(e^{BA})$. By the theorem of Cayley and Hamilton, this implies that $(q(e^{BA}))^m = (p(e^{BA}))^{km} = 0$. Thus the matrix $q(e^{BA})$ is nilpotent, too.

2.8 Solutions of Olympic 2001

2.8.1 Day 1

Problem 1. Since there are exactly n rows and n columns, the choice is of the form

$$\{(j, \sigma(j) : \ j = 1, \dots, n\}$$

where $\sigma \in S_n$ is a permutation. Thus the corresponding sum is equal to

$$\sum_{j=1}^{n} n(j-1) + \sigma(j) = \sum_{j=1}^{n} nj - \sum_{j=1}^{n} n + \sum_{j=1}^{n} \sigma(j)$$

$$= n\sum_{j=1}^{n} j - \sum_{j=1}^{n} n + \sum_{j=1}^{n} j = (n+1)\frac{n(n+1)}{2} - n^2 = \frac{n(n^2+1)}{2}.$$

which shows that the sum is independent of σ.

Problem 2.

1. There exist integers u and v such that $us + vt = 1$. Since $ab = ba$, we obtain

$$ab = (ab)^{us+vt} = (ab)^{us}((ab)^t)^v = (ab)^{us}e = (ab)^{us} = a^{us}(b^s)^u = a^{us}e = a^{us}.$$

Therefore, $b^r = eb^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e$. Since $xr+ys = 1$ for suitable integers x and y,

$$b = b^{xr+ys} = (b^r)^x (b^s)^y = e.$$

It follows similarly that $a = e$ as well.

2. This is not true. Let $a = (123)$ and $b = (34567)$ be cycles of the permutation group S_7 of order 7. Then $ab = (1234567)$ and $a^3 = b^5 = (ab)^7 = e$.

Problem 3.

$$\lim_{t\to 1-0}(1-t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n} = \lim_{t\to 1-0}\frac{1-t}{-\ln t}(-\ln t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n} =$$

$$= \lim_{t\to 1-0}(-\ln t)\sum_{n=1}^{\infty}\frac{1}{1+e^{-n\ln t}} = \lim_{h\to +0} h\sum_{n=1}^{\infty}\frac{1}{1+e^{nh}} = \int_0^{\infty}\frac{dx}{1+e^x} = \ln 2.$$

Problem 4. Let $p(x)=(x-1)^k r(x)$ and $\epsilon_j = e^{1\pi ij/q}$ $(j=1,2,\dots,q-1)$. As is well-known, the polynomial $x^{q-1}+x^{q-2}+\dots+x+1=(x-\epsilon_1)\dots(x-\epsilon_{q-1})$ is irreducible, thus all $\epsilon_1,\dots,\epsilon_{q-1}$ are roots of $r(x)$, or none of them.

Suppose that none of $\epsilon_1,\dots,\epsilon_{q-1}$ is a root of $r(x)$. Then $\prod_{j=1}^{q-1} r(\epsilon_j)$ is a rational integer, which is not 0 and

$$(n+1)^{q-1} \ge \prod_{j=1}^{q-1} |p(\epsilon_j)| = \left|\prod_{j=1}^{q-1}(1-\epsilon_j)^k\right| \left|\prod_{j=1}^{q-1} r(\epsilon_j)\right| \ge$$

$$\ge \left|\prod_{j=1}^{q-1}(1-\epsilon_j)\right|^k = (1^{q-1}+1^{q-2}+\dots+1^1+1)^k = q^k.$$

This contradicts the condition $\dfrac{q}{\ln q} < \dfrac{k}{\ln(n+1)}$.

Problem 5.

The statement will be proved by induction on n. For $n=1$, there is nothing to do. In the case $n=2$, write $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$. If $b\neq 0$, and $c\neq 0$ or $b=c=0$ then A is similar to

$$\begin{bmatrix} 1 & 0 \\ a/b & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 1 & 0 \\ -a/b & 1\end{bmatrix} = \begin{bmatrix} 0 & b \\ c-ad/b & a+d\end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -a/c \\ 0 & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 1 & a/c \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & b-ad/c \\ c & a+d\end{bmatrix}$$

respectively. If $b=c=0$ and $a\neq d$, then A is similar to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & d-a \\ 0 & d\end{bmatrix},$$

and we can perform the step seen in the case $b\neq 0$ again.

Assume now that $n>3$ and the problem has been solved for all $n'<n$. Let $A=\begin{bmatrix} A' & * \\ * & \beta\end{bmatrix}_n$, where A' is $(n-1)\times(n-1)$ matrix. Clearly we may assume that $A'\neq \lambda' I$, so the induction provides a P with, say, $P^{-1}A'P=\begin{bmatrix} 0 & * \\ * & \alpha\end{bmatrix}_{n-1}$. But then the matrix

$$B=\begin{bmatrix} P^{-1} & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} A' & * \\ * & \beta\end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta\end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \ldots, 0, \alpha, \beta)$. On the other hand, we may also view B as $\begin{bmatrix} 0 & * \\ * & C \end{bmatrix}$, where C is an $(n-1) \times (n-1)$ matrix with diagonal $(0, \ldots, 0, \alpha, \beta)$. If the inductive hypothesis is applicable to C, we would have $Q^{-1}CQ = D$, with $D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_{n-1}$ so that finally the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} .B. \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \ldots, 0, \gamma)$, as required.

The inductive argument can fail only when $n-1 = 2$ and the resulting matrix applying P has the form

$$P^{-1}AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix}$$

where $d \neq 0$. The numbers a, b, c, e cannot be 0 at the same time. If, say, $b \neq 0$, A is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ c & d & 0 \\ e-b-d & a & b+d \end{bmatrix}.$$

Performing half of the induction step again, the diagonal of the resulting matrix will be $(0, d-b, d+b)$ (the trace is the same) and the induction step can be finished. The cases $a \neq 0, c \neq 0$ and $e \neq 0$ are similar.

Problem 6.

Let $0 < \epsilon < A$ be an arbitrary real number. If x is sufficiently large then $f(x) > 0, g(x) > 0, |a(x) - A| < \epsilon, |b(x) - B| < \epsilon$ and

$$\begin{aligned} B - \epsilon < b(x) = \frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} < \frac{f'(x)}{g'(x)} + (A+\epsilon)\frac{f(x)}{g(x)} < \\ < \frac{(A+\epsilon)(A+1)}{A} \cdot \frac{f'(x)(g(x))^A + A.f(x).(g(x))^{A-1}.g'(x)}{(A+1).(g(x))^A.g'(x)} = \\ = \frac{(A+\epsilon)(A+1)}{A} \cdot \frac{(f(x).(g(x))^A)'}{((g(x))^{A+1})'}, \end{aligned} \tag{1}$$

thus

$$\frac{(f(x).(g(x))^A)'}{((g(x))^{A+1})'} > \frac{A(B-\epsilon)}{(A+\epsilon)(A+1)}. \tag{2}$$

It can be similarly obtained that, for sufficiently large x,

$$\frac{(f(x).(g(x))^A)'}{((g(x))^{A+1})'} < \frac{A(B+\epsilon)}{(A-\epsilon)(A+1)}. \tag{3}$$

From $\epsilon \to 0$, we have

$$\lim_{x\to\infty} \frac{(f(x).(g(x))^A)'}{((g(x))^{A+1})'} = \frac{B}{A+1}.$$

By l'Hospital's rule this implies

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f(x).(g(x))^A}{(g(x))^{A+1}} = \frac{B}{A+1}.$$

2.8.2 Day 2

Problem 1.

Multiply the left hand side polynomials. We obtain the following equalities:

$$a_0b_0 = 1,\ a_0b_1 + a_1b_0 = 1, \ldots$$

Among them one can find equations

$$a_0 + a_1b_{s-1} + a_2b_{s-2} + \cdots = 1$$

and

$$b_0 + b_1a_{r-1} + b_2a_{r-2} + \cdots = 1.$$

From these equations it follows that $a_0, b_0 \leq 1$. Taking into account that $a_0b_0 = 1$ we can see that $a_0 = b_0 = 1$.

Now looking at the following equations we notice that all $a's$ must be less than or equal to 1. The same statement holds for the $b's$. It follows from $a_0b_1 + a_1b_0 = 1$ that one of the numbers a_1, b_1 equals 0 while the other one must be 1. Follow by induction.

Problem 2. Obviously $a_2 = \sqrt{2-\sqrt{2}} < \sqrt{2}$.

Since the function $f(x) = \sqrt{2-\sqrt{4-x^2}}$ is increasing on the interval $[0, 2]$ the inequality $a_1 > a_2$ implies that $a_2 > a_3$. Simple induction ends the proof of monotonicity of (a_n). In the same way we prove that (b_n) decreases (just notice that

$$g(x) = \frac{2x}{2+\sqrt{4+x^2}} = \frac{2}{\frac{2}{x}+\sqrt{1+\frac{4}{x^2}}}.$$

It is a matter of simple manipulation to prove that $2f(x) > x$ for all $x \in (0,2)$, this implies that the sequence $(2^n a_n)$ is strictly increasing. The inequality $2g(x) < x$ for $x \in (0,2)$ implies that the sequence $(2^n b_n)$ strictly decreases. By an easy induction one can show that $a_n^2 = \dfrac{4b^2}{4+b_n^2}$ for positive integers n. Since the limit of the decreasing sequence $(2^n b_n)$ of positive numbers is finite we have

$$\lim 4^n a_n^2 = \lim \frac{4.4^n b_n^2}{4+b_n^2} = \lim 4^n b_n^2.$$

We know already that the limits $\lim 2^n a_n$ and $\lim 2^n b_n$ are equal. The first of the two is positive because the sequence $(2^n a_n)$ is strictly increasing. The existence of a number C follows easily from the equalities

$$2^n b_n - 2^n a_n = \left(4^n b_n^2 - \frac{4^{n+1} b_n^2}{4+b_n^2}\right)/(2^n b_n + 2^n a_n) = \frac{(2^n b_n)^4}{4+b_n^2}\cdot\frac{1}{4^n}\cdot\frac{1}{2^n(b_n+a_n)}$$

and from the existence of positive limits $\lim 2^n b_n$ and $\lim 2^n a_n$.

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of sine and cosine. It is enough to notice that $a_n = 2\sin\dfrac{\pi}{2^{n+1}}$ and $b_n = 2tan\dfrac{\pi}{2^{n+1}}$.

Problem 3.

The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{(x_1,\ldots,x_n) \in \mathbb{R}^n \middle| \sum_{k=1}^n x_k^2 = 1\right\}.$$

The distance between the points $X = (x_1,\ldots,x_n)$ and $Y = (y_1,\ldots,y_n)$ is:

$$d^2(X,Y) = \sum_{k=1}^n (x_k - y_k)^2.$$

We have

$$\begin{aligned} d(X,Y) > \sqrt{2} &\Leftrightarrow d^2(X,Y) > 2 \\ &\Leftrightarrow \sum_{k=1}^n x_k^2 + \sum_{k=1}^n y_k^2 + 2\sum_{k=1}^n x_k y_k > 2 \\ &\Leftrightarrow \sum_{k=1}^n x_k y_k < 0 \end{aligned}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1, 0, \ldots, 0).$$

For $X = A_1$, $\sum\limits_{k=1}^n x_k y_k < 0$ implies $y_1 > 0$, $\forall Y \in M_n$.

Let $X = (x_1, \overline{X})$, $Y = (y_1, \overline{Y}) \in M_n \backslash \{A_1\}$, $\overline{X}, \overline{Y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^n x_k y_k < 0 \ \Rightarrow \ x_1 y_1 + \sum_{k=1}^{n-1} \overline{x_k y_k} < 0 \ \Leftrightarrow \ \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\overline{x_k}}{\sqrt{\sum \overline{x_k}^2}}, \ y'_k = \frac{\overline{y_k}}{\sqrt{\sum \overline{y_k}^2}}.$$

therefore

$$(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$$

and verifies $\sum\limits_{k=1}^n x_k y_k < 0$.

If a_n is the search number of points in $\mathbb{R}^n$ we obtain $a_n \leq 1 + a_{n-1}$ and $a_1 = 2$ implies that $a_n \leq n+1$.

We show that $a_n = n+1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$\begin{aligned} A_1 &= (-1, 0, 0, 0, \ldots, 0, 0) \\ A_2 &= \left(\frac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right) \\ A_3 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, -c_2, 0, \ldots, 0, 0\right) \\ A_4 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-1} \cdot c_2, -c_3, \ldots, 0, 0\right) \end{aligned}$$

$$A_{n-1} = \Big(\frac{1}{n}, \frac{1}{n-1}\cdot c_1, \frac{1}{n-2}\cdot c_2, \frac{1}{n-3}\cdot c_3, \dots, -c_{n-2}, 0\Big)$$
$$A_n = \Big(\frac{1}{n}, \frac{1}{n-1}\cdot c_1, \frac{1}{n-2}\cdot c_2, \frac{1}{n-3}\cdot c_3, \dots, \frac{1}{2}\cdot c_{n-2}, -c_{n-1}\Big)$$
$$A_{n+1} = \Big(\frac{1}{n}, \frac{1}{n-1}\cdot c_1, \frac{1}{n-2}\cdot c_2, \frac{1}{n-3}\cdot c_3, \dots, \frac{1}{2}\cdot c_{n-2}, c_{n-1}\Big)$$

where

$$c_k = \sqrt{\Big(1+\frac{1}{n}\Big)\Big(1-\frac{1}{n-k+1}\Big)},\ k = \overline{1, n-1}.$$

We have $\sum\limits_{k=1}^{n} x_k y_k = -\dfrac{1}{n} < 0$ and $\sum\limits_{k=1}^{n} x_k^2 = 1,\ \forall X, Y \in \{A_1, \dots, A_{n+1}\}$. These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

$$d = \sqrt{2}\sqrt{1+\frac{1}{n}} > \sqrt{2}.$$

Remark. For $n = 2$ the points form an equilateral triangle in the unit circle; for $n = 3$ the four points from a regular tetrahedron and in $\mathbb{R}^n$ the points from an n dimensional regular simplex.

Problem 4.

We will only prove (2), since it implies (1). Consider a directed graph G with n vertices $V_1, \dots, V_n$ and a directed edge from V_k to V_l when $a_{k,l} \neq 0$. We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length m. Let $j_1 < \dots < j_m$ be the vertices the cycle goes through and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k, j_{\sigma_0}(k)} \neq 0$ for $k = 1, \dots, m$. Observe that for any other $\sigma \in S_n$ we have $a_{j_k, j_{\sigma(k)}} = 0$ for some $k \in \{1, \dots, m\}$, otherwise we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally

$$\begin{aligned} 0 &= det(a_{j_k, j_l})_{k,l=1,\dots,m} \\ &= (-1)^{sign\sigma_0} \prod_{k=1}^{m} a_{j_k, j_{\sigma_0}(k)} + \sum_{\sigma\neq\sigma_0} (-1)^{sign\sigma} \prod_{k=1}^{m} a_{j_k, j_\sigma(k)} \neq 0, \end{aligned}$$

which is a contradiction.

Since G is acyclic there exists a topological ordering i.e. a permutation $(\sigma \in S_n$ such that $k < l$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(l)}$. It is easy to see that this permutation solves the problem.

Problem 5. Suppose that there exists a function satisfying the inequality. If $f(f(x)) \leq 0$ for all x, then f is a decreasing function in view of the inequalities $f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$ for any $y \leq 0$. Since $f(0) > 0 \geq f(f(x))$, it implies $f(x) > 0$ for all x, which is a contradiction. Hence there is a z such that $f(f(z)) > 0$. Then the inequality $f(z+x) \geq f(z) + xf(f(z))$ shows that $\lim\limits_{x\to\infty} f(x) = +\infty$ and therefore $\underset{x\to\infty}{f}(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that $f(x) \geq 0, f(f(x)) > 1, y \geq \dfrac{x+1}{f(f(x))-1}$ and $f(f(x+y+1)) \geq 0$. Then $f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1$ and hence

$$\begin{aligned} f(f(x+y)) &\geq f(x+y+1) + (f(x+y) - (x+y+1))f(f(x+y+1)) \geq \\ &\geq f(x+y+1) \geq f(x+y) + f(f(x+y)) \geq \\ &\geq f(x) + yf(f(x)) + f(f(x+y)) > f(f(x+y)). \end{aligned}$$

This contradiction completes the solution of the problem.

Problem 6. We prove that $g(\vartheta) = |\sin\vartheta||\sin(2\vartheta)|^{1/2}$ attains its maximum value $(\frac{\sqrt{3}}{2})^{3/2}$ at points $\frac{2^k\pi}{3}$ (where k is a positive integer). This can be seen by using derivatives or a classical bound like

$$\begin{aligned} |g(\vartheta)| = |\sin\vartheta||\sin(2\vartheta)|^{1/2} &= \frac{\sqrt{2}}{\sqrt[4]{3}}\Big(\sqrt[4]{|\sin\vartheta|.|\sin\vartheta|.|\sin\vartheta|.|\sqrt{3}\cos\vartheta|}\Big)^2 \\ &\leq \frac{\sqrt{2}}{\sqrt[4]{3}} \cdot \frac{3\sin^2\vartheta + 3\cos^2\vartheta}{4} = \Big(\frac{\sqrt{3}}{2}\Big)^{3/2}. \end{aligned}$$

Hence

$$\begin{aligned} &\Big|\frac{f_n(\vartheta)}{f_n(\pi/3)}\Big| = \\ &= \Big|\frac{g(\vartheta).g(2\vartheta)^{1/2}.g(4\vartheta)^{3/4}\cdots g(2^{n-1}\vartheta)^E}{g(\pi/3).g(2\pi/3)^{1/2}.g(4\pi/3)^{3/4}\cdots g(2^{n-1}\pi/3)^E}\Big| \cdot \Big|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\Big|^{1-E/2} \\ &\leq \Big|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\Big|^{1-E/2} \leq \Big(\frac{1}{\sqrt{3}/2}\Big)^{1-E/2} \leq \frac{2}{\sqrt{3}}. \end{aligned}$$

where $E = \frac{2}{3}(1 - (-\frac{1}{2})^n)$. This is exactly the bound we had to prove.

2.9 Solutions of Olympic 2002

2.9.1 Day 1

Problem 1. First we show that the standard parabola with vertex V contains point A if and only if the standard parabola with vertex $s(A)$ contains point $s(V)$.

Let $A = (a, b)$ and $V = (v, w)$. The equation of the standard parabola with vertex $V = (v, w)$ is $y = (x - v)^2 + w$, so it contains point A if and only if $b = (a - v)^2 + w$. Similarly, the equation of the parabola with vertex $s(A) = (a, -b)$ is $y = (x-a)^2-b$; it contains point $s(V) = (v, -w)$ if and only if $-w = (v - a)^2 - b$. The two conditions are equivalent. Now assume that the standard parabolas with vertices V_1 and V_2, V_1 and V_3, V_2 and V_3 intersect each other at points A_3, A_2, A_1, respectively. Then, by the statement above, the standard parabolas with vertices $s(A_1)$ and $s(A_2), S(A_1)$ and $s(A_3), s(A_2)$ and $S(A_3)$ intersect each other at points V_3, V_2, V_1, respectively, because they contain these points.

Problem 2. Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly monotone increasing.

By the monotonity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus, $f(0)$ is a lower bound for $f'(x)$, and for all $x < 0$ we have $f(x) < f(0) + x.f(0) = (1+x)f(0)$. Hence, if $x \le -1$ then $f(x) \le 0$, contradicting the property $f(x) > 0$.

So such function does not exist.

Problem 3. Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \; b_k = 2^{k-n}, \text{ for } k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} = 0. \tag{1}$$

Solution. Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for all $k \ge 1$, (1) is equivalent to

$$\frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right] = \frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}. \tag{2}$$

We prove (2) by induction. For $n=1$, both sides are equal to 2.

Assume that (2) holds for some n. Let

$$x_n = \frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right];$$

then

$$x_{n+1} = \frac{2^{n+1}}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{2^n}{n+1}\left(1+\sum_{k=0}^{n-1}\left(\frac{1}{\binom{n}{k}}+\frac{1}{\binom{n}{k+1}}\right)+1\right) =$$

$$= \frac{2^n}{n+1}\sum_{k=0}^{n-1}\frac{\frac{n-k}{n}+\frac{k+1}{n}}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}.$$

This implies (2) for $n+1$.

Problem 4. If for some $n > m$ the equality $p_m = p_n$ holds then T_p is a finite set. Thus we can assume that all points $p_0, p_1, \ldots$ are distinct. There is a convergent subsequence p_{n_k} and its limit q is in T_p. Since f is continuous $p_{n_k+1} = f(p_{n_k}) \to f(q)$, so all, except for finitely many, points p_n are accumulation points of T_p. Hence we may assume that all of them are accumulation points of T_p. Let $d = \sup\{|p_m - p_n| : m, n \geq 0\}$. Let δ_n be positive numbers such that $\sum\limits_{n=0}^{\infty}\delta_n < \frac{d}{2}$. Let I_n be an interval of length less than δ_n centered at p_n such that there are there are infinitely many $k's$ such that $p_k \notin \bigcup\limits_{j=0}^{n} I_j$, this can be done by induction. Let $n_0 = 0$ and n_{m+1} be the smallest integer k ¿ nm such that $p_k \notin \bigcup\limits_{j=0}^{n_m} I_j$. Since T_p is closed the limit of the subsequence (p_{n_m}) must be in T_p but it is impossible because of the definition of $I_n's$, of course if the sequence (p_{n_m}) is not convergent we may replace it with its convergent subsequence. The proof is finished.

Remark. If $T_p = \{p_1, p_2, \ldots\}$ and each p_n is an accumulation point of T_p, then T_p is the countable union of nowhere dense sets (i.e. the single-element sets $\{p_n\}$). If T is closed then this contradicts the Baire Category Theorem.

Problem 5.

a. It does not exist. For each y the set $\{x : y = f(x)\}$ is either empty or consists of 1 point or is an interval. These sets are pairwise disjoint, so there are at most count ably many of the third type.

b. Let f be such a map. Then for each value y of this map there is an xo such that $y = f(x)$ and $f'(x) = 0$, because an uncountable set $\{x : y = f(x)\}$ contains an accumulation point x_0 and clearly $f'(x_0) = 0$. For every $\epsilon > 0$ and every x_0 such that $f'(x_0) = a$ there exists an open interval I_{x_0} such that if $x \in I_{x_0}$ then $|f'(x)| < \epsilon$. The union of all these intervals I_{x_0} may be written as a union of pairwise disjoint open intervals J_n. The image of each J_n is an interval (or a point) of length $< \epsilon$ length (J_n) due to Lagrange Mean Value Theorem. Thus the image of the interval $[0, 1]$ may be covered with the intervals such that the sum of their lengths is $\epsilon.1 = \epsilon$. This is not possible for $\epsilon < 1$.

Remarks. 1. The proof of part **b** is essentially the proof of the easy part of A. Sard's theorem about measure of the set of critical values of a smooth map.

2. If only continuity is required, there exists such a function, e.g. the first co-ordinate of the very well known Peano curve which is a continuous map from an interval onto a square.

Problem 6.

Lemma 1. Let $(a_n)_{n\geq 0}$ be a sequence of non-negative numbers such that $a_{2k} - a_{2k+1} \leq a_k^2$, $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1}$ for any $k \geq 0$ and $\limsup n a_n < \frac{1}{4}$. Then $\limsup \sqrt[n]{a_n} < 1$.

Proof. Let $c_l = \sup_{n\geq 2^l}(n+1)a_n$ for $l \geq 0$. We will show that $c_{l+1} \leq 4c_l^2$. Indeed, for any integer $n \geq 2^{l+1}$ there exists an integer $k \geq 2^l$ such that $n = 2k$ or $n = 2k+1$. In the first case there is $a_{2k} - a_{2k+1} \leq a_k^2 \leq \frac{c_l^2}{(k+1)^2} \leq \frac{4c_l^2}{2k+1} - \frac{4c_l^2}{2k+2}$, whereas in the second case there is $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1} \leq \frac{c_l^2}{(k+1)(k+2)} \leq \frac{4c_l^2}{2k+2} - \frac{4c_l^2}{2k+3}$.

Hence a sequence $(a_n - \frac{4c_l^2}{n+1})_{n\geq 2^{l+1}}$ is non-decreasing and its terms are non-positive since it converges to zero. Therefore $a_n \le \frac{4c_l^2}{n+1}$ for $n \ge 2^{l+1}$, meaning that $c_{l+1}^2 \le 4c_l^2$. This implies that a sequence $((4c_l)^{2^{-l}})_{l\ge0}$ is nonincreasing and therefore bounded from above by some number $q \in (0,1)$ since all its terms except finitely many are less than 1. Hence $c_l \le q^{2^l}$ for l large enough. For any n between 2^l and 2^{l+1} there is $a_n \le \frac{c_l}{n+1} \le q^{2^l} \le (\sqrt{q})^n$ yielding $\limsup \sqrt[n]{a_n} \le \sqrt{q} < 1$, yielding $\limsup \sqrt[n]{a_n} \le \sqrt{1} < 1$, which ends the proof.

Lemma 2. Let T be a linear map from $\mathbb{R}^n$ into itself. Assume that $\limsup n \parallel T^{n+1} - T^n \parallel < \frac{1}{4}$. Then $\limsup \parallel T^{n+1} - T^n \parallel^{1/n} < 1$. In particular T^n converges in the operator norm and T is power bounded.

Proof. Put $a_n = \parallel T^{n+1} - T^n \parallel$. Observe that

$$T^{k+m+1} - T^{k+m} = (T^{k+m+2} - T^{k+m+1}) - (T^{k+1} - T^k)(T^{m+1} - T^m)$$

implying that $a_{k+m} \le a_{k+m+1} + a_k a_m$. Therefore the sequence $(a_m)_{m\ge0}$ satisfies assumptions of Lemma 1 and the assertion of Proposition 1 follows.

Remarks. 1. The theorem proved above holds in the case of an operator T which maps a normed space X into itself, X does not have to be finite dimensional.

2. The constant $\frac{1}{4}$ in Lemma 1 cannot be replaced by any greater number since a sequence $a_n = \frac{1}{4n}$ satisfies the inequality $a_{k+m} - a_{k+m+1} \le a_k a_m$ for any positive integers k and m whereas it does not have exponential decay.

3. The constant $\frac{1}{4}$ in Lemma 2 cannot be replaced by any number greater that $\frac{1}{e}$. Consider an operator $(Tf)(x) = xf(x)$ on $L^2([0,1])$. One can easily check that $\limsup \parallel T^{n+1} - T^n \parallel = \frac{1}{e}$, whereas T^n does not converge in the operator norm. The question whether in general $\limsup n \parallel T^{n+1} - T^n \parallel < \infty$ implies that T is power bounded remains open.

Remark. The problem was incorrectly stated during the competition: instead of the inequality $\| A^k - A^{k-1} \| \leq \dfrac{1}{2002k}$, the inequality $\| A^k - A^{k-1} \| \leq \dfrac{1}{2002n}$ was assumed. If $A = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$ then $A^k = \begin{pmatrix} 1 & k\epsilon \\ 0 & 1 \end{pmatrix}$. Therefore $A^k - A^{k-1} = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$, so for sufficiently small ϵ the condition is satisfied although the sequence $(\| A^k \|)$ is clearly unbounded.

2.9.2 Day 2

Problem 1. Adding the second row to the first one, then adding the third row to the second one, ..., adding the nth row to the $(n-l)th$, the determinant does not change and we have

$$det(A) = \begin{vmatrix} 2 & -1 & +1 & \dots & \pm 1 & \mp 1 \\ -1 & 2 & -1 & \dots & \mp 1 & \pm 1 \\ +1 & -1 & 2 & \dots & \pm 1 & \mp 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp 1 & \pm 1 & \mp 1 & \dots & 2 & -1 \\ \pm 1 & \mp 1 & \pm 1 & \dots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \pm 1 & \mp 1 & \pm 1 & \mp 1 & \dots & -1 & 2 \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting second column from the third, ..., and at last, subtract the $(n-1)th$ column from the nth column. This way we have

$$det(A) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n+1 \end{vmatrix} = n+1.$$

Problem 2. For each pair of students, consider the set of those problems which was not solved by them. There exist $\binom{200}{2} = 19900$ sets; we have to prove that at least one set is empty.

For each problem, there are at most 80 students who did not solve it. From these students at most $\binom{80}{2} = 3160$ pairs can be selected, so the problem can belong to at most 3160 sets. The 6 problems together can belong to at most $6.3160 = 18960$ sets.

Hence, at least $19900 - 18960 = 940$ sets must be empty.

Problem 3. We prove by induction on n that $\frac{a_n}{e}$ and $b_n e$ are integers, we prove this for $n = 0$ as well. (For $n = 0$, the term 0^0 in the definition of the sequences must be replaced by 1.) From the power series of e^x, $a_n = e^1 = e$ and $b_n = e^{-1} = \frac{1}{e}$.

Suppose that for some $n \geq 0, a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are all multipliers of e and $\frac{1}{e}$, respectively. Then, by the binomial theorem,

$$a_{n+1} = \sum_{k=0}^{n} \frac{(k+1)^{n+1}}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{k!} = \sum_{k=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!} =$$
$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} \frac{k^m}{k!} = \sum_{m=0}^{n} \binom{n}{m} a_m$$

and similarly

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{k+1} \frac{(k+1)^{n+1}}{(k+1)!} = -\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)^n}{k!}$$
$$= -\sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!}$$
$$= -\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k \frac{k^m}{k!} = -\sum_{m=0}^{n} \binom{n}{m} b_m.$$

The numbers a_{n+1} and b_{n+1} are expressed as linear combinations of the previous elements with integer coefficients which finishes the proof.

Problem 4. We can assume $OA = OB = OC = 1$. Intersect the unit sphere with center O with the angle domains AOB, BOC and COA; the intersections are "slices" and their areas are $\frac{1}{2}\gamma$, $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$, respectively.

Now project the slices AOC and COB to the plane OAB. Denote by C' the projection of vertex C, and denote by A' and B' the reflections of vertices A and B with center 0, respectively. By the projection, $OC' < 1$.

The projections of arcs AC and BC are segments of ellipses with long axes AA' and BB', respectively. (The ellipses can be degenerate if σ or τ is right angle.) The two ellipses intersect each other in 4 points; both half ellipses connecting A and A' intersect both half ellipses connecting

B and B'. There exist no more intersection, because two different conics cannot have more than 4 common points.

The signed areas of the projections of slices AOC and COB are $\frac{1}{2}\alpha.\cos\tau$ and $\frac{1}{2}\beta.\cos\sigma$ respectively. The statement says thet the sum of these signed areas is less than the area of slice BOA.

There are three significantly different cases with respect to the signs of $\cos\sigma$ and $\cos\tau$ (see Figure). If both signs are positive (case (a)), then the projections of slices OAC and OBC are subsets of slice OBC without common interior point, and they do not cover the whole slice OBC; this implies the statement. In cases (b) and (c) where at least one of the signs is negative, projections with positive sign are subsets of the slice OBC, so the statement is obvious again.

Problem 5. The direction $\Leftarrow$ is trivial, since if $A = S\overline{S}^{-1}$, then $A\overline{A} = S\overline{S}^{-1} = I_n$.

For the direction $\Rightarrow$, we must prove that there exists an invertible matrix S such that $A\overline{S} = S$.

Let w be an arbitrary complex number which is not 0. Choosing $S = wA + \overline{w}I_n$ we have $A\overline{S} = A(\overline{w}\overline{A} + wI_n) = \overline{w}I_n + wA = S$. If S is singular, then $\frac{1}{w}S = A - \left(\frac{\overline{w}}{w}\right)I_n$ is singular as well, so $\frac{\overline{w}}{w}$ is an eigenvalue of A. Since A has finitely many eigenvalues and $\frac{\overline{w}}{w}$ can be any complex number on the unit circle, there exist such w that S is invertible.

Problem 6. Let $g(x) = f(x) - f(x_1) - < \nabla f(x_1), x - x_1 >$. It is obvious that g has the same properties. Moreover, $g(x_1) = \nabla g(x_1) = 0$ and, due to convexity, g has 0 as the absolute minimum at x_1. Next we prove that

$$g(x_2) \geq \frac{1}{2L} \parallel \nabla g(x_2) \parallel^2 . \tag{2}$$

Let $y_0 = x_2 - \dfrac{1}{L} \parallel \nabla g(x_2) \parallel$ and $y(t) = y_0 + t(x_2 - y_0)$. Then

$$g(x_2) = g(y_0) + \int\limits_0^1 < \nabla g(y(t)), x_2 - y_0 > dt =$$

$$= g(y_0) + < \nabla g(x_2), x_2 - y_0 > - \int\limits_0^1 < \nabla g(x_2) - \nabla g(y(t)), x_2 - y_0 > dt \geq$$

$$\geq 0 + \frac{1}{L} \parallel \nabla g(x_2) \parallel^2 - \int\limits_0^1 \parallel \nabla g(x_2) - \nabla g(y(t)) \parallel . \parallel x_2 - y_0 \parallel dt \geq$$

$$\geq \frac{1}{L} \parallel \nabla g(x_2) \parallel^2 - \parallel x_2 - y_0 \parallel \int\limits_0^1 L \parallel x_2 - g(y) \parallel dt =$$

$$= \frac{1}{L} \parallel \nabla g(x_2) \parallel^2 - L \parallel x_2 - y_0 \parallel^2 \int\limits_0^1 t dt = \frac{1}{2L} \parallel \nabla g(x_2) \parallel^2 .$$

Substituting the definition of g into (2), we obtain

$$f(x_2) - f(x_1) - < \nabla f(x_1), x_2 - x_1 > \geq \frac{1}{2L} \parallel \nabla f(x_2) - \nabla f(x_1) \parallel^2,$$

$$\parallel \nabla f(x_2) - \nabla f(x_1) \parallel^2 \leq 2L < \nabla f(x_1), x_1 - x_2 > + 2L(f(x_2) - f(x_1)). \tag{3}$$

Exchanging variables x_1 and x_2, we have

$$\parallel \nabla f(x_2) - \nabla f(x_1) \parallel^2 \leq 2L < \nabla f(x_2), x_2 - x_1 > + 2L(f(x_1) - f(x_2)). \tag{4}$$

The statement (1) is the average of (3) and (4).

2.10 Solutions of Olympic 2003

2.10.1 Day 1

Problem 1.

a) Let $b_n = \dfrac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$. Then $a_{n+1} > \dfrac{3}{2} a_n$ is equivalent to $b_{n+1} > b_n$, thus the sequence (b_n) is strictly increasing. Each increasing sequence has a finite limit or tends to infinity.

b) For all $\alpha > 1$ there exists a sequence $1 = b_1 < b_2 < \dots$ which converges to α. Choosing $a_n = \left(\frac{3}{2}\right)^{n-1} b_n$, we obtain the required sequence (a_n).

Problem 2. Let $S = a_1 + a_2 + \cdots + a_{51}$. Then $b_1 + b_2 + \cdots + b_{51} = 50S$. Since $b_1, b_2, \dots, b_{51}$ is a permutation of $a_1, a_2, \dots, a_{51}$, we get $50S = S$, so $49S = 0$. Assume that the characteristic of the field is not equal to 7. Then $49S = 0$ implies that $S = 0$. Therefore $b_i = -a_i$ for $i = 1, 2, \dots, 51$. On the other hand, $b_i = a_{\varphi(i)}$, where $\varphi \in S_{51}$. Therefore, if the characteristic is not 2, the sequence $a_1, a_2, \dots, a_{51}$ can be partitioned into pairs $\{a_i, a_{\varphi(i)}\}$ of additive inverses. But this is impossible, since 51 is an odd number. It follows that the characteristic of the field is 7 or 2.

The characteristic can be either 2 or 7. For the case of 7, $x_1 = \cdots = x_{51} = 1$ is a possible choice. For the case of 2, any elements can be chosen such that $S = 0$, since then $b_i = -a_i = a_i$.

Problem 3. The minimal polynomial of A is a divisor of $3x^3 - x^2 - x - 1$. This polynomial has three different roots. This implies that A is diagonalizable: $A = C^{-l}DC$ where D is a diagonal matrix. The eigenvalues of the matrices A and D are all roots of polynomial $3x^3 - x^2 - x - 1$. One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of D^k, which are the kth powers of the eigenvalues, tend to either 0 or 1 and the limit $M = \lim D^k$ is idempotent. Then $\lim A^k = C^{-1}MC$ is idempotent as well.

Problem 4. Clearly a and b must be different since A and B are disjoint.

Let $\{a, b\}$ be a solution and consider the sets A, B such that $a.A = b.B$. Denoting $d = (a, b)$ the greatest common divisor of a and b, we have $a = d.a_1, b = d.b_1, (a_1, b_1) = 1$ and $a_1 A = b_1 B$. Thus $\{a_1, b_1\}$ is a solution and it is enough to determine the solutions $\{a, b\}$ with $(a, b) = 1$.

If $1 \in A$ then $a \in a.A = b.B$, thus b must be a divisor of a. Similarly, if $1 \in B$, then a is a divisor of b. Therefore, in all solutions, one of numbers a, b is a divisor of the other one.

Now we prove that if $n \geq 2$, then $(1, n)$ is a solution. For each positive integer k, let $f(k)$ be the largest non-negative integer for which $n^{f(k)}|k$. Then let $A = \{k : f(k) \text{ is odd}\}$ and $B = \{k : f(k) \text{ is even}\}$. This is a decomposition of all positive integers such that $A = n.B$.

Problem 5.

B. We shall prove in two different ways that $\lim_{n\to\infty} f_n(x) = g(0)$ for every $x \in (0, 1]$. (The second one is more lengthy but it tells us how to calculate f_n directly from g.)

Proof I. First we prove our claim for non-decreasing g. In this case, by induction, one can easily see that

1. each f_n is non-decrasing as well, and
2. $g(x) = f_0(x) \geq f_1(x) \geq f_2(x) \geq \cdots \geq g(0)$ $(x \in (0, 1])$.

Then (2) implies that there exists

$$h(x) = \lim_{n\to\infty} f_n(x) \quad (x \in (0, 1]).$$

Clearly h is non-decreasing and $g(0) \leq h(x) \leq f_n(x)$ for any $x \in (0, 1], n = 0, 1, 2, \ldots$. Therefore to show that $h(x) = g(0)$ for any $x \in (0, 1]$, it is enough to prove that $h(1)$ cannot be greater than $g(0)$.

Suppose that $h(1) > g(0)$. Then there exists a $0 < \delta < 1$ such that $h(1) > g(\delta)$. Using the definition, (2) and (1) we get

$$f_{n+1}(1) = \int_0^1 f_n(t)dt \leq \int_0^\delta g(t)dt + \int_\delta^1 f_n(t)dt \leq \delta g(\delta) + (1 - \delta)f_n(1).$$

Hence

$$f_n(1) - f_{n+1}(1) \geq \delta(f_n(1) - g(\delta)) \geq \delta(h(1) - g(\delta)) > 0,$$

so $f_n(1) \to -\infty$, which is a contradiction.

Similarly, we can prove our claim for non-increasing continuous functions as well.

Now suppose that g is an arbitrary continuous function on $[0, 1]$. Let

$$M(x) = \sup_{t\in[0,x]} g(t), \quad m(x) = \inf_{t\in[0,x]} g(t) \quad (x \in [0, 1])$$

Then on $[0,1]$ m is non-increasing, M is non-decreasing, both are continuous, $m(x) \le g(x) \le M(x)$ and $M(0) = m(0) = g(0)$. Define the sequences of functions $M_n(x)$ and $m_n(x)$ in the same way as f_n is defined but starting with $M_0 = M$ and $m_0 = m$.

Then one can easily see by induction that $m_n(x) \le f_n(x) \le M_n(x)$. By the first part of the proof, $\lim_n m_n(x) = m(0) = g(0) = M(0) = \lim_n M_n(x)$ for any $x \in (0,1]$. Therefore we must have $\lim_n f_n(x) = g(0)$.

Proof II. To make the notation clearer we shall denote the variable of f_j by x_j. By definition (and Fubini theorem) we get that

$$\begin{aligned} f_{n+1}(x_{n+1}) &= \frac{1}{x_{n+1}} \int_0^{x_{n+1}} \frac{1}{x_n} \int_0^{x_n} \frac{1}{x_{n-1}} \int_0^{x_{n-1}} \cdots \int_0^{x_2} \frac{1}{x_1} \int_0^{x_1} g(x_0) dx_0 dx_1 \dots dx_n \\ &= \frac{1}{x_{n+1}} \int \cdots \int_{0 \le x_0 \le x_1 \le \cdots \le x_n \le x_{n+1}} g(x_0) \frac{dx_0 dx_1 \dots dx_n}{x_1 \dots x_n} \\ &= \frac{1}{x_{n+1}} \int_0^{x_{n+1}} g(x_0) \Big(\int \cdots \int_{x_0 \le x_1 \le \cdots \le x_n \le x_{n+1}} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \Big) dx_0. \end{aligned}$$

Therefore with the notation

$$h_n(a,b) = \int \cdots \int_{a \le x_1 \le \cdots \le x_n \le b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$

and $x = x_{n+1}, t = x_0$ we have

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) h_n(t,x) dt.$$

Using that $h_n(a,b)$ is the same for any permutation of $x_1, \dots, x_n$ and the fact that the integral is 0 on any hyperplanes $(x_i = x_j)$ we get that

$$\begin{aligned} n! h_n(a,b) &= \int \cdots \int_{a \le x_1, \dots, x_n \le b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} = \int_a^b \cdots \int_a^b \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \\ &= \Big(\int_a^b \frac{dx}{x} \Big)^n = \Big(\log \frac{b}{a} \Big)^n. \end{aligned}$$

Therefore

$$f_{n+1}(x) = \frac{1}{x}\int_0^x g(t)\frac{(\log(x/t))^n}{n!}dt.$$

Note that if g is constant then the definition gives $f_n = g$. This implies on one hand that we must have

$$\frac{1}{x}\int_0^x \frac{(\log(x/t))^n}{n!}dt = 1$$

and on the other hand that, by replacing g by $g - g(0)$, we can suppose that $g(0) = 0$.

Let $x \in (0,1]$ and $\epsilon > 0$ be fixed. By continuity there exists a $0 < \delta < x$ and an M such that $|g(t)| < \epsilon$ on $[0,\delta]$ and $|g(t)| \leq M$ on $[0,1]$. Since

$$\lim_{n\to\infty} \frac{(\log(x/\delta))^n}{n!} = 0$$

there exists an n_0 sucht that $\dfrac{\log(x/\delta)^n}{n!} < \epsilon$ whenever $n \geq n_0$. Then, for any $n \geq n_0$, we have

$$\begin{aligned}
|f_{n+1}(x)| &\leq \frac{1}{x}\int_0^x |g(t)|\frac{(\log(x/t))^n}{n!}dt \\
&\leq \frac{1}{x}\int_0^\delta \epsilon\frac{(\log(x/t))^n}{n!}dt + \frac{1}{x}\int_\delta^x |g(t)|\frac{(\log(x/\delta))^n}{n!}dt \\
&\leq \frac{1}{x}\int_0^x \epsilon\frac{(\log(x/t))^n}{n!}dt + \frac{1}{x}\int_\delta^x M\epsilon dt \\
&\leq \epsilon + M\epsilon.
\end{aligned}$$

Therefore $\lim_n f(x) = 0 = g(0)$.

Problem 6. The polynomial f is a product of linear and quadratic factors, $f(z) = \prod_i(k_i z + l_i).\prod_j(p_j z^2 + q_j z + r_j)$, with $k_i, k_i, p_j, q_j \in \mathbb{R}$. Since all roots are in the left half-plane, for each i, k_i and l_i are of the same sign, and for each j, p_j, q_j, r_j are of the same sign, too. Hence, multiplying f by -1 if necessary, the roots of f don't change and f becomes the polynomial with all positive coefficients.

For the simplicity, we extend the sequence of coefficients by $a_{n+1} = a_{n+2} = \cdots = 0$ and $a_{-1} = a_{-2} = \cdots = 0$ and prove the same statement for $-1 \le k \le n-2$ by induction.

For $n \le 2$ the statement is obvious: a_{k+1} and a_{k+2} are positive and at least one of a_{k-1} and a_{k+3} is 0; hence, $a_{k+1}a_{k+2} > a_k a_{k+3} = 0$.

Now assume that $n \ge 3$ and the statement is true for all smaller values of n. Take a divisor of $f(z)$ which has the form z^2+pz+q where p and q are positive real numbers. (Such a divisor can be obtained from a conjugate pair of roots or two real roots.) Then we can write

$$f(z) = (z^2+pz+q)(b_{n-2}z^{n-2} + \cdots + b_1 z + b_0) = (z^2+pz+q)g(x). \quad (1)$$

The roots polynomial $g(z)$ are in the left half-plane, so we have $b_{k+1}b_{k+2} < b_k b_{k+3}$ for all $-1 \le k \le n-4$. Defining $b_{n-1} = b_n = \cdots = 0$ and $b_{-1} = b_{-2} = \cdots = 0$ as well, we also have $b_{k+1}b_{k+2} \le b_k b_{k+3}$ for all integer k.

Now we prove $a_{k+1}a_{k+2} > a_k a_{k+3}$. If $k = -1$ or $k = n-2$ then this is obvious since $a_{k+1}a_{k+2}$ is positive and $a_k a_{k+3} = 0$. Thus, assume $0 \le k \le n-3$. By an easy computation,

$$\begin{gathered}
a_{k+1}a_{k+2} - a_k a_{k+3} = \\
= (qb_{k+1} + pb_k + b_{k-1})(qb_{k+2} + pb_{k+1} + b_k) - \\
-(qb_k + pb_{k-1} + b_{k-2})(qb_{k+3} + pb_{k+2} + b_{k+1}) = \\
= (b_{k-1}b_k - b_{k-2}b_{k+1}) + p(b_k^2 - b_{k-2}b_{k+2}) + q(b_{k-1}b_{k+2} - b_{k-2}b_{k+3}) + \\
+p^2(b_k b_{k+1} - b_{k-1}b_{k+2}) + q^2(b_{k+1}b_{k+2} - b_k b_{k+3}) + pq(b_{k+1}^2 - b_{k-1}b_{k+3}).
\end{gathered}$$

We prove that all the six terms are non-negative and at least one is positive. Term $p^2(b_k b_{k+1} - b_{k-1}b_{k+2}$ is positive since $0 \le k \le n-3$. Also terms $b_{k-1}b_k - b_{k-2}b_{k+1}$ and $q^2(b_{k+1}b_{k+2} - b_k b_{k+3})$ are non-negative by the induction hypothesis.

To check the sign of $p(b_k^2 - b_{k-2}b_{k+2})$ consider

$$b_{k-1}(b_k^2 - b_{k-2}b_{k+2})$$

$$= b_{k-2}(b_k b_{k+1} - b_{k-1}b_{k+2}) + b_k(b_{k-1}b_k - b_{k-2}b_{k+1}) \geq 0.$$

If $b_{k-1} > 0$ we can divide by it to obtain $b_k^2 - b_{k-2}b_{k+2} \geq 0$. Otherwise, if $b_{k-1} = 0$, either $b_{k-2} = 0$ or $b_{k+2} = 0$ and thus $b_k^2 - b_{k-2}b_{k+2} = b_k^2 \geq 0$. Therefore, $p(b_k^2 - b_{k-2}b_{k+2}) \geq 0$ for all k. Similarly, $pq(b_{k+1}^2 - b_{k-1}b_{k+3}) \geq 0$.

The sign of $q(b_{k-1}b_{k+2} - b_{k-2}b_{k+3})$ can be checked in a similar way. Consider

$$b_{k+1}(b_{k-1}b_{k+2} - b_{k-2}b_{k+3}) = b_{k-1}(b_{k+1}b_{k+2} - b_k b_{k+3}) + b_{k+3}(b_{k-1}b_k - b_{k-2}b_{k+1}) \geq 0.$$

If $b_{k+1} > 0$, we can divide by it. Otherwise either $b_{k-2} = 0$ or $b_{k+3} = 0$. In all cases, we obtain $b_{k-1}b_{k+2} - b_{k-2}b_{k+3} \geq 0$.

Now the signs of all terms are checked and the proof is complete.

2.10.2 Day 2

Problem 1. We use the fact that $\dfrac{\sin t}{t}$ is decreasing in the interval $(0, \pi)$ and $\lim\limits_{t \to 0+0} \dfrac{\sin t}{t} = 1$. For all $x \in (0, \frac{\pi}{2})$ and $t \in [x, 2x]$ we have $\dfrac{\sin 2x}{2}x < \dfrac{\sin t}{t} < 1$, thus

$$\left(\frac{\sin 2x}{2x}\right)^m \int_x^{2x} \frac{t^m}{t^n} < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} \frac{t^m}{t^n} dt,$$

$$\int_x^{2x} \frac{t^m}{t^n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

The factor $\left(\dfrac{\sin 2x}{2x}\right)^m$ tends to 1. If $m-n+1 < 0$, the limit of x^{m-n+1} is infinity; if $m-n+1 > 0$ then 0. If $m-n+1 = 0$ then $x^{m-n+1} \int_1^2 u^{m-n} du =$

$\ln 2$. Hence,

$$\lim_{x\to 0+0}\int_{x}^{2x}\frac{\sin^m t}{t^n}dt=\begin{cases}0, & m\ge n\\ \ln 2, & n-m=1\\ +\infty, & n-m>1.\end{cases}$$

Problem 3. Let $b_0 \notin A$ (otherwise $b_0 \in A \subset B$, $\varrho = \inf_{a\in A}|a-b_0|$. The intersection of the ball of radius $\varrho+1$ with centre b_0 with set A is compact and there exists $a_0 \in A : |a_0 - b_0| = \varrho$.

Denote by $\mathbf{B}_r(a) = \{x \in R^n : |x-a| \le r\}$ and $\partial\mathbf{B}_r(a) = \{x \in R^n : |x-a| = r\}$ the ball and the sphere of center a and radius r, respectively.

If a_0 is not the unique nearest point then for any point a on the open line segment (a_0, b_0) we have $\mathbf{B}_{|a-a_0|}(a) \subset \mathbf{B}_\varrho(b_0)$ and $\partial\mathbf{B}_{|a-a_0|}(a) \cap \partial\mathbf{B}_\varrho(b_0) = \{a_0\}$, therefore $(a_0, b_0) \subset B$ and b_0 is an accumulation point of set B.

Problem 4. The condition (i) of the problem allows us to define a (well-defined) operation * on the set S given by

$$a * b = c \text{ if and only if } \{a, b, c\} \in F, \text{ where } a \ne b.$$

We note that this operation is still not defined completely (we need to define $a * a$), but nevertheless let us investigate its features. At first, due to (i), for $a \ne b$ the operation obviously satisfies the following three conditions:

a) $a \ne a * b \ne b$;

b) $a * b = b * a$;

c) $a * (a * b) = b$.

What does the condition (ii) give? It claims that

e') $x*(a*c) = x*y = z = b*c = (x*a)*c$ for any three different x, a, c, i.e. that the operation is associative if the arguments are different. Now we can complete the definition of *. In order to save associativity for nondifferent arguments, i.e. to make $b = a * (a * b) = (a * a) * b$ hold, we will add to S an extra element, call it 0, and define

d) $a * a = 0$ and $a * 0 = 0 * a = a$.

Now it is easy to check that, for any $a, b, c \in S \cup \{0\}$, (a),(b),(c) and (d), still hold, and

e) $a * b * c := (a * b) * c = a * (b * c)$.

We have thus obtained that $(S \cup \{0\}, *)$ has the structure of a finite Abelian group, whose elements are all of order two. Since the order of every such group is a power of 2, we conclude that $|S \cup \{0\}| = n+1 = 2^m$ and $n = 2^m - 1$ for some integer $m \geq 1$.

Given $n = 2^m - 1$, according to what we have proven till now, we will construct a family of three-element subsets of S satisfying (i) and (ii). Let us define the operation * in the following manner:

if $a = a_0 + 2a_1 + \cdots + 2^{m-1}a_{m-1}$ and $b = b_0 + 2b_1 + \cdots + 2^{m-1}b_{m-1}$, where a_i, b_i are either 0 or 1, we put $a * b = |a_0 - b_0| + 2|a_1 - b_1| + \cdots + 2^{m-1}|a_{m-1} - b_{m-1}|$.

It is simple to check that this * satisfies (a),(b),(c) and (e'). Therefore, if we include in F all possible triples $a, b, a * b$, the condition (i) follows from (a),(b) and (c), whereas the condition (ii) follows from (e')

The answer is: $n = 2^m - 1$.

Problem 5.

a) Let $\varphi : \mathbb{Q} \to \mathbb{N}$ be a bijection. Define $g(x) = \max\{|f(s,t)| : s, t \in \mathbb{Q},\ \varphi(s) \leq \varphi(x), \varphi(t) \leq \varphi(x)\}$. We have $f(x,y) \leq \max\{g(x), g(y)\} \leq g(x) + g(y)$.

b) We shall show that the function defined by $f(x,y) = \dfrac{1}{|x-y|}$ for $x \neq y$ and $f(x,x) = 0$ satisfies the problem. If, by contradiction there exists a function g as above, it results, that $g(y) \geq \dfrac{1}{|x-y|} - fx)$ for $x, y \in \mathbb{R}, x \neq y$; one obtains that for each $x \in \mathbb{R}$, $\lim\limits_{y \to x} g(y) = \infty$. We show, that there exists no function g having an infinite limit at each point of a bounded and closed interval $[a,b]$. For each $k \in \mathbb{N}^+$ denote $A_k = \{x \in [a,b] : |g(x)| \leq k\}$.

We have obviously $[a,b] = \bigcup\limits_{k=1}^{\infty} A_k$. The set $[a,b]$ is uncountable, so at least one of the sets A_k is infinite (in fact uncountable). This set

A_k being infinite, there exists a sequence in A_k having distinct terms. This sequence will contain a convergent subsequence $(x_n)_{n\in\mathbb{N}}$ convergent to a point $x \in [a,b]$. But $\lim\limits_{y\to x} g(y) = \infty$ implies that $g(xn) \to \infty$, a contradiction because $|g(xn)| \le k, \forall n \in \mathbb{N}$.

Second solution for part (b). Let S be the set of all sequences of real numbers. The cardinality of S is $|S| = |\mathbb{R}|^{\mathcal{N}_0} = 2^{\mathcal{N}_0^2} = 2^{\mathcal{N}_0} = |\mathbb{R}|$. Thus, there exists a bijection $h : \mathbb{R} \to S$. Now define the function f in the following way. For any real x and positive integer n, let $f(x,n)$ be the nth element of sequence $h(x)$. If y is not a positive integer then let $f(x,y) = 0$. We prove that this function has the required property.

Let g be an arbitrary $\mathbb{R} \to \mathbb{R}$ function. We show that there exist real numbers x, y such that $f(x,y) > g(x)+g(y)$. Consider the sequence $(n+g(n))_{n=1}^{\infty}$. This sequence is an element of S, thus $(n+g(n))_{n=1}^{\infty} = h(x)$ for a certain real x. Then for an arbitrary positive integer n, $f(x,n)$ is the nth element, $f(x,n) = n+g(n)$. Choosing n such that $n > g(x)$, we obtain $f(x,n) = n + g(n) > g(x) + g(n)$.

Problem 6. Consider the generating function $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. By induction $0 < a_n \le 1$, thus this series is absolutely convergent for $|x| < 1$, $f(0) = 1$ and the function is positive in the interval $[0,1)$. The goal is to compute $f(\frac{1}{2})$.

By the recurrence formula,

$$f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a_k}{n-k+2}x^n =$$
$$= \sum_{k=0}^{\infty} a_k x^k \sum_{n=k}^{\infty}\frac{x^{n-k}}{n-k+2} = f(x)\sum_{m=0}^{\infty}\frac{x^m}{m+2}.$$

Then

$$\ln f(x) = \ln f(x) - \ln f(0) = \int\limits_0^x \frac{f'}{f} = \sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)(m+2)} =$$

$$= \sum_{m=1}^{\infty} \Big(\frac{x^{m+1}}{(m+1} - \frac{x^{m+1}}{(m+2)}\Big) = 1 + \Big(1 - \frac{1}{x}\Big) \sum_{m=1}^{\infty} \frac{x^{m+1}}{(m+1)} = 1 + \Big(1 - \frac{1}{x}\Big) \ln \frac{1}{1-x},$$

$$\ln f\Big(\frac{1}{x}\Big) = 1 - \ln 2,$$

and thus $f\Big(\frac{1}{2}\Big) = \frac{e}{2}$.

2.11 Solutions of Olympic 2004

2.11.1 Day 1

Problem 1. Let $S_n = S \cap \left(\frac{1}{n}, \infty\right)$ for any integer $n > 0$. It follows from the inequality that $|S_n| < n$. Similarly, if we define $S_{-n} = S \cap (-\infty, -\frac{1}{n})$, then $|S_{-n}| < n$. Any nonzero $x \in S$ is an element of some S_n or S_{-n}, because there exists an n such that $x > \dfrac{1}{n}$ or $x < -\dfrac{1}{n}$. Then $S \subset \{0\} \cup \bigcup\limits_{n \in N} (S_n \cup S_{-n}), S$ is a countable union of finite sets, and hence countable.

Problem 2. Put $P_n(x) = \underbrace{P(P(\ldots(P}_{n}(x))\ldots))$. As $P_1(x) \geq -1$, for each $x \in R$, it must be that $P_{n+1}(x) = P_1(P_n(x)) \geq -1$, for each $n \in N$ and each $x \in R$. Therefore the equation $P_n(x) = a$, where $a < -1$ has no real solutions.

Let us prove that the equation $P_n(x) = a$, where $a > 0$, has exactly two distinct real solutions. To this end we use mathematical induction by n. If $n = 1$ the assertion follows directly. Assuming that the assertion holds for a $n \in N$ we prove that it must also hold for $n + 1$. Since $P_{n+1}(x) = a$ is equivalent to $P_1(P_n(x)) = a$, we conclude that $P_n(x) = \sqrt{a+1}$ or $P_n(x) = -\sqrt{a+1}$. The equation $P_n(x) = \sqrt{a+1}, as \sqrt{a+1} > 1$, has exactly two distinct real solutions by the inductive hypothesis, while the equation $P_n(x) = -\sqrt{a+1}$ has no real solutions (because

$-\sqrt{a+1} < -1$). Hence the equation $P_{n+1}(x) = a$, has exactly two distinct real solutions.

Let us prove now that the equation $P_n(x) = 0$ has exactly $n+1$ distinct real solutions. Again we use mathematical induction. If $n = 1$ the solutions are $x = \pm 1$, and if $n = 2$ the solutions are $x = 0$ and $x = \pm\sqrt{2}$, so in both cases the number of solutions is equal to $n+1$. Suppose that the assertion holds for some $n \in N$. Note that $P_{n+2}(x) = P_2(P_n(x)) = P_n^2(x)(P_n^2(x) - 2)$, so the set of all real solutions of the equation $P_{n+2} = 0$ is exactly the union of the sets of all real solutions of the equations $P_n(x) = 0, P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$. By the inductive hypothesis the equation $P_n(x) = 0$ has exactly $n+1$ distinct real solutions, while the equations $P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$ have two and no distinct real solutions, respectively. Hence, the sets above being pairwise disjoint, the equation $P_{n+2}(x) = 0$ has exactly $n+3$ distinct real solutions. Thus we have proved that, for each $n \in N$, the equation $P_n(x) = 0$ has exactly $n+1$ distinct real solutions, so the answer to the question posed in this problem is 2005.

Problem 3.

a) Equivalently, we consider the set

$$Y = \{y = (y_1, y_2, \ldots, y_n) | 0 \le y_1, y_2, \ldots, y_n \le 1,\ y_1 + y_2 + \cdots + y_n = 1\} \subset R^n$$

and the image $f(Y)$ of Y under

$$f(y) = \arcsin y_+ \arcsin y_2 + \cdots + \arcsin y_n.$$

Note that $f(Y) = S_n$. Since Y is a connected subspace of R^n and f is a continuous function, the image $f(Y)$ is also connected, and we know that the only connected subspaces of R are intervals. Thus S^n is an interval.

b) We prove that

$$n \arcsin \frac{1}{n} \le x_1 + x_2 + \cdots + x_n \le \frac{\pi}{2}.$$

Since the graph of $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$, the chord joining the points $(0,0)$ and $(\frac{\pi}{2}, 1)$ lies below the graph. Hence

$$\frac{2x}{\pi} \leq \sin x \text{ for all } x \in [0, \frac{\pi}{2}]$$

and we can deduce the right-hand side of the claim:

$$\frac{2}{\pi}(x_1 + x_2 + \cdots + x_n) \leq \sin x_1 + \sin x_2 + \cdots + \sin x_n = 1.$$

The value 1 can be reached choosing $x_1 = \frac{\pi}{2}$ and $x_2 = \cdots = x_n = 0$.

The left-hand side follows immediately from Jensen's inequality, since $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$ and $0 \leq \frac{x_1 + x_2 + \cdots + x_n}{n} < \frac{\pi}{2}$

$$\frac{1}{n} = \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{n} \leq \sin \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Equality holds if $x_1 = \cdots = x_n = \arcsin \frac{1}{n}$.

Now we have computed the minimum and maximum of interval S_n; we can conclude that $S_n = [n \arcsin \frac{1}{n}, \frac{\pi}{2}]$. Thus $l_n = \frac{\pi}{2} - n$ and

$$\lim_{n\to\infty} l_n = \frac{\pi}{2} - \lim_{n\to\infty} \frac{\arcsin(1/n)}{1/n} = \frac{\pi}{2} - 1.$$

Problem 4. Define $f : M \to \{-1, 1\}$, $f(X) = \begin{cases} -1, & \text{if } X \text{ is white} \\ 1, & \text{if } X \text{ is black} \end{cases}$. The given condition becomes $\sum\limits_{X \in S} f(X) = 0$ for any sphere S which passes through at least 4 points of M. For any 3 given points A, B, C in M, denote by $S(A, B, C)$ the set of all spheres which pass through A, B, C and at least one other point of M and by $|S(A, B, C)|$ the number of these spheres. Also, denote by $\sum$ the sum $\sum\limits_{X \in M} f(X)$.

We have

$$0 = \sum_{S \in S(A,B,C)} \sum_{X \in S} f(X) = (|S(A, B, C)| - 1)(f(A) + f(B) + f(C)) + \sum \tag{1}$$

since the values of A, B, C appear $|S(A, B, C)|$ times each and the other values appear only once.

If there are 3 points A, B, C such that $|S(A, B, C)| = 1$, the proof is finished.

If $|S(A, B, C)| > 1$ for any distinct points A, B, C in M, we will prove at first that $\sum = 0$.

Assume that $\sum > 0$. From (1) it follows that$f(A)+f(B)+f(C) < 0$ and summing by all $\binom{n}{3}$ possible choices of (A, B, C) we obtain that $\binom{n}{3}\sum < 0$, which means $\sum < 0$ (contradicts the starting assumption). The same reasoning is applied when assuming $\sum < 0$.

Now, from $\sum = 0$ and (1), it follows that $f(A) + f(B) + f(C) = 0$ for any distinct points A, B, C in M. Taking another point $D \in M$, the following equalities take place

$$\begin{aligned} f(A) + f(B) + f(C) &= 0 \\ f(A) + f(B) + f(D) &= 0 \\ f(A) + f(C) + f(D) &= 0 \\ f(B) + f(C) + f(D) &= 0 \end{aligned}$$

which easily leads to $f(A) = f(B) = f(C) = f(D) = 0$, which contradicts the definition of f.

Problem 5. We prove a more general statement:

Lemma. Let $k, l \geq 2$, let X be a set of $\binom{k+l-4}{k-2}$ real numbers. Then either X contains an increasing sequence $\{x_i\}_{i=1}^k \subseteq X$ of length k and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \ \forall i = 2, \ldots, k-1,$$

or X contains a decreasing sequence $\{x_i\}_{i=1}^l \subseteq X$ of length l and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \ \forall i = 2, \ldots, l-1.$$

Proof of the lemma. We use induction on $k+l$. In case $k = 2$ or $l = 2$ the lemma is obviously true.

Now let us make the induction step. Let m be the minimal element of X, M be its maximal element. Let

$$X_m = \{x \in X : \; x \le \frac{m+M}{2}\}, \; X_M = \{x \in X : \; x > \frac{m+M}{2}\}.$$

Since $\binom{k+l-4}{k-2} = \binom{k+(l-1)-4}{k-2} + \binom{(k-1)+l-4}{(k-1)-2}$, we can see that either

$$|X_m| \ge \binom{(k-1)+l-4}{(k-1)-2} + 1, \text{ or } |X_M| \ge \binom{k+(l-1)-4}{k-2} + 1.$$

In the first case we apply the inductive assumption to X_m and either obtain a decreasing sequence of length l with the required properties (in this case the inductive step is made), or obtain an increasing sequence $\{x_i\}_{i=1}^{k-1} \subseteq X_m$ of length $k-1$. Then we note that the sequence $\{x_1, x_2, \ldots, x_{k-1}, M\} \subseteq X$ has length k and all the required properties.

In the case $|X_M| \ge \binom{k+(l-1)-4}{k-2}$ the inductive step is made in a similar way. Thus the lemma is proved.

The reader may check that the number $\binom{k+l-4}{k-2} + 1$ cannot be smaller in the lemma.

Problem 6. It is clear that the left hand side is well defined and independent of the order of summation, because we have a sum of the type $\sum n^{-4}$, and the branches of the logarithms do not matter because all branches are taken. It is easy to check that the convergence is locally uniform on $\mathbb{C}\backslash\{0,1\}$; therefore, f is a holomorphic function on the complex plane, except possibly for isolated singularities at 0 and 1. (We omit the detailed estimates here.)

The function log has its only (simple) zero at $z=1$, so f has a quadruple pole at $z=1$.

Now we investigate the behavior near infinity. We have $Re(\log(z)) =$

$\log|z|$, hence (with $c := \log|z|$)

$$
\begin{aligned}
|\sum(\log z)^{-4}| \le \sum|\log z|^{-4} &= \sum(\log|z| + 2\pi i n)^{-4} + O(1)\\
&= \int_{-\infty}^{\infty}(x + 2\pi i x)^{-4}dx + O(1)\\
&= c^{-4}\int_{-\infty}^{\infty}(1 + 2\pi i x/c)^{-4}dx + O(1)\\
&= c^{-3}\int_{-\infty}^{\infty}(1 + 2\pi i t)^{-4}dt + O(1)\\
&\le \alpha(\log|z|)^{-3}
\end{aligned}
$$

for a universal constant α. Therefore, the infinite sum tends to 0 as $|z| \to \infty$. In particular, the isolated singularity at ∞ is not essential, but rather has (at least a single) zero at ∞.

The remaining singularity is at $z = 0$. It is readily verified that $f(1/z) = f(z)$ (because $\log(1/z) = -\log(z)$); this implies that f has a zero at $z = 0$.

We conclude that the infinite sum is holomorphic on $\mathbb{C}$ with at most one pole and without an essential singularity at ∞, so it is a rational function, i.e. we can write $f(z) = P(z)/Q(z)$ for some polynomials P and Q which we may as well assume coprime. This solves the first part.

Since j has a quadruple pole at $z = 1$ and no other poles, we have $Q(z) = (z-1)^4$ up to a constant factor which we can as well set equal to 1, and this determines P uniquely. Since $f(z) \to 0$ as $z \to \infty$, the degree of P is at most 3, and since $P(0) = 0$, it follows that $P(z) = z(az^2+bz+c)$ for yet undetermined complex constants a, b, c.

There are a number of ways to compute the coefficients a, b, c, which turn out to be $a = c = \frac{1}{6}, b = \frac{2}{3}$. Since $f(z) = f(\frac{l}{z})$, it follows easily that $a = c$. Moreover, the fact $\lim_{z\to 1}(z-1)^4 f(z) = 1$ implies $a + b + c = 1$ (this fact follows from the observation that at $z = 1$, all summands cancel pairwise, except the principal branch which contributes a quadruple pole). Finally, we can calculate

$$f(-1) = \pi^{-4} \sum_{n odd} n^{-4} = 2\pi^{-4} \sum_{n \geq 1 odd} n^{-4}$$

$$= 2\pi^{-4} \Big(\sum_{n \geq 1} n^{-4} - \sum_{n \geq 1 even} n^{-4} \Big) = \frac{1}{48}.$$

This implies $a - b + c = -\frac{1}{3}$. These three equations easily yield a, b, c.

Moreover, the function f satisfies $f(z) + f(-z) = 16 f(z^2)$: this follows because the branches of $\log(z^2) = \log((-z)^2)$ are the numbers $2\log(z)$ and $2\log(-z)$. This observation supplies the two equations $b = 4a$ and $a = c$, which can be used instead of some of the considerations above.

Another way is to compute $g(z) = \sum \frac{1}{(\log z)^2}$ first. In the same way, $g(z) = \frac{dz}{(z-1)^2}$. The unknown coefficient d can be computed from $\lim_{z \to 1} (z-1)^2 g(z) = 1$; it is $d = 1$. Then the exponent 2 in the denominator can be increased by taking derivatives (see Solution 2). Similarly, one can start with exponent 3 directly.

A more straightforward, though tedious way to find the constants is computing the first four terms of the Laurent series of f around $z = 1$. For that branch of the logarithm which vanishes at 1, for $|w| < \frac{1}{2}$ we have

$$\log(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + O(|w^5|);$$

after some computation, one can obtain

$$\frac{1}{\log(1 + w)^4} = w^{-4} + 2w^{-2} + \frac{7}{6} w^{-2} + \frac{1}{6} w^{-1} + O(1).$$

The remaining branches of logarithm give a bounded function. So

$$f(1 + w) = w^{-4} + 2w^{-2} + \frac{7}{6} w^{-2} + \frac{1}{6} w^{-1}$$

(the remainder vanishes) and

$$f(z) = \frac{1 + 2(z-1) + \frac{7}{6}(z-1)^2 + \frac{1}{6}(z-1)^3}{(z-1)^4} = \frac{z(z^2 + 4z + 1)}{6(z-1)^4}.$$

Solution 2. From the well-known series for the cotangent function,

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{w+2\pi i.k}=\frac{i}{2}\cot\frac{iw}{2}$$

and

$$\lim_{n\to\infty}\sum_{k=-N}^{N}\frac{1}{\log z+2\pi i.k}=\frac{i}{2}\cot\frac{i\log z}{2}=\frac{i}{2}.i\frac{2^{2i\frac{i\log z}{2}}+1}{2^{2i\frac{i\log z}{2}}-1}=\frac{1}{2}+\frac{1}{z-1}.$$

Taking derivatives we obtain

$$\sum\frac{1}{(\log z)^2}=-z.\Big(\frac{1}{2}+\frac{1}{z-1}\Big)'=\frac{z}{(z-1)^2},$$

$$\sum\frac{1}{(\log z)^3}=-\frac{z}{2}.\Big(\frac{1}{(z-1)^2}\Big)'=\frac{z(z+1)}{2(z-1)^3}$$

and

$$\sum\frac{1}{(\log z)^4}=-\frac{z}{3}\Big(\frac{z(z+1)}{2(z-1)^3}\Big)'=\frac{z(z^2+4z+1)}{2(z-1)^4}.$$

2.11.2 Day 2

Problem 1. Let $A=\binom{A_1}{A_2}$ and $B=(B_1B_2)$ where A_1,A_2,B_1,B_2 are 2×2 matrices. Then

$$\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}=\begin{pmatrix}A_1\\A_2\end{pmatrix}(B_1\ B_2)=\begin{pmatrix}A_1B_1&A_1B_2\\A_2B_1&A_2B_2\end{pmatrix}$$

therefore, $A_1B_1=A_2B_2=I_2$ and $A_1B_2=A_2B_1=-I_2$ Then $B_1=A_1^{-1},B_2=-A_1^{-1}$ and $A_2=B_2^{-1}=-A_1$. Finally,

$$BA=(B_1\ \ B_2)\begin{pmatrix}A_1\\A_2\end{pmatrix}=B_1A_1+B_2A_2=2I_2=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

Problem 2. Let $F(x)=\int\limits_a^x\sqrt{f(t)}dt$ and $G(x)=\int\limits_a^x\sqrt{g(t)}dt$. The functions F,G are convex, $F(a)=0=G(a)$ and $F(b)=G(b)$ by the hypothesis. We are supposed to show that

$$\int\limits_a^b\sqrt{1+(F'(t))^2}dt\geq\int\limits_a^b\sqrt{1+(G'(t))^2}dt$$

i.e. The length at the graph of F is $\geq$ the length of the graph of G. This is clear since both functions are convex, their graphs have common ends and the graph of F is below the graph of G - the length of the graph of F is the least upper bound of the lengths of the graphs of piecewise linear functions whose values at the points of non-differentiability coincide with the values of F, if a convex polygon P_1 is contained in a polygon P_2 then the perimeter of P_1 is $\leq$ the perimeter of P_2.

Problem 3. Considering as vectors, thoose p to be the unit vector which points into the opposite direction as $\sum\limits_{i=1}^{n} p_i$. Then, by the triangle inequality,

$$\sum_{i=1}^{n} |p - p_i| \geq \left| np - \sum_{i=1}^{n} p_i \right| = n + \left| \sum_{i=1}^{n} p_i \right| \geq n.$$

Problem 4. We first solve the problem for the special case when the eigenvalues of M are distinct and all sums $\lambda_r + \lambda_s$ are different. Let λ_r and λ_s be two eigenvalues of M and $\overrightarrow{v}_r$, $\overrightarrow{v}_s$ eigenvectors associated to them, i.e. $M\overrightarrow{v}_j = \lambda \overrightarrow{v}_j$ for $j = r, s$. We have $M\overrightarrow{v}_r(\overrightarrow{v}_s)^T + \overrightarrow{v}_r(\overrightarrow{v}_s)^T M^T = (M\overrightarrow{v}_r)(\overrightarrow{v}_s)^T + \overrightarrow{v}_r(M\overrightarrow{v}_s)^T = \lambda_r \overrightarrow{v}_r(\overrightarrow{v}_s)^T + \lambda_s \overrightarrow{v}_r(\overrightarrow{v}_s)^T$, so $\overrightarrow{v}_r(\overrightarrow{v}_s)$ is an eigenmatrix of L_M with the eigenvalue $\lambda_r + \lambda_s$.

Notice that if $\lambda_r \neq \lambda_s$ then vectors $\overrightarrow{u}$, $\overrightarrow{w}$, ware linearly independent and matrices $\overrightarrow{u}(\overrightarrow{w})^T$ and $\overrightarrow{w}(\overrightarrow{u}^T$ are linearly independent, too. This implies that the eigenvalue $\lambda_r + \lambda_s$ is double if $r \neq s$.

The map L_M maps n^2-dimensional linear space into itself, so it has at most n^2 eigenvalues. We already found n^2 eigenvalues, so there exists no more and the problem is solved for the special case.

In the general case, matrix M is a limit of matrices $M_1, M_2, \ldots$ such that each of them belongs to the special case above. By the continuity of the eigenvalues we obtain that the eigenvalues of L_M are

- $2\lambda_r$ with multiplicity m_r^2 $(r = 1, \ldots, k)$;
- $\lambda_r + \lambda_s$ with multiplicity $2m_r m_s (1 \leq r < s \leq k)$.

(It can happen that the sums $\lambda_r + \lambda_s$ are not pairwise different; for those multiple values the multiplicities should be summed up.)

Problem 5. First we use the inequality

$$x^{-1} - 1 \geq |\ln x|,\ x \in (0,1],$$

which follows from

$$(x^{-1}-1)|_{x=1} = |\ln x||_{x=1} = 0,$$
$$(x^{-1}-1)' = -\frac{1}{x^2} \leq -\frac{1}{x} = |\ln x|',\ x \in (0,1].$$

Therefore

$$\int_0^1\int_0^1 \frac{dxdy}{x^{-1}+|\ln y|-1} \leq \int_0^1\int_0^1 \frac{dxdy}{|\ln x|+|\ln y|} = \int_0^1\int_0^1 \frac{dxdy}{|\ln(x.y)|}.$$

Substituting $y = \dfrac{u}{x}$, we obtain

$$\int_0^1\int_0^1 \frac{dxdy}{|\ln(x.y)|} = \int_0^1 \Big(\int_u^1 \frac{dx}{x}\Big)\frac{du}{|\ln u|} = \int_0^1 |lnu|\frac{du}{|\ln u|} = 1.$$

Solution 2. Substituting $s = x^{-1} - 1$ and $u = s - \ln y$,

$$\int_0^1\int_0^1 \frac{dxdy}{x^{-1}+|\ln y|-1} = \int_0^\infty\int_s^\infty \frac{e^{s-u}}{(s+1)^2 u}duds = \int_0^\infty \Big(\int_0^u \frac{e^s}{(s+1)^2}ds\Big)\frac{e^{-u}}{u}dsdu.$$

Since the function $\dfrac{e^s}{(s+1)^2}$ is convex,

$$\int_0^u \frac{e^s}{(s+1)^2}ds \leq \frac{u}{2}\Big(\frac{e^u}{(u+1)^2}+1\Big)$$

so

$$\int_0^1\int_0^1 \frac{dxdy}{x^{-1}+|\ln y|-1} \leq \int_0^\infty \frac{u}{2}\Big(\frac{e^u}{(u+1)^2}+1\Big)\frac{e^{-u}}{u}du$$

$$= \frac{1}{2}\Big(\int\limits_0^\infty \frac{du}{(u+1)^2} + \int\limits_0^\infty e^{-u}du\Big) = 1.$$

Problem 6. The quantity $S(A_n^{k-1})$ has a special combinatorical meaning. Consider an $n \times k$ table filled with 0's and 1's such that no 2×2 contains only 1's. Denote the number of such fillings by F_{nk}. The filling of each row of the table corresponds to some integer ranging from 0 to $2^n - 1$ written in base 2. F_{nk} equals to the number of k-tuples of integers such that every two consecutive integers correspond to the filling of $n \times 2$ table without 2×2 squares filled with 1's.

Consider binary expansions of integers i and $j\overline{i_n i_{n-1} \dots i_1}$ and $\overline{j_n j_{n-1} \dots j_1}$. There are two cases:

1. If $i_n j_n = 0$ then i and j can be consecutive iff $\overline{i_{n-1} \dots i_1}$ and $\overline{j_{n-1} \dots j_1}$ can be consequtive.

2. If $i_n = j_n = 1$ then i and j can be consecutive iff $i_{n-1} j_{n-1} = 0$ and $\overline{i_{n-2} \dots i_1}$ and $\overline{j_{n-2} \dots j_1}$ can be consecutive.

Hence i and j can be consecutive iff $(i+1, j+1)$-th entry of A_n is 1. Denoting this entry by $a_{i,j}$, the sum $S(A_n^{k-1}) = \sum\limits_{i_1=0}^{2^n-1} \cdots \sum\limits_{i_k=0}^{2^n-1} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k}$ counts the possible fillings. Therefore $F_{nk} = S(A_n^{k-1})$.

The the obvious statement $F_{nk} = F_{kn}$ completes the proof.

2.12 Solutions of Olympic 2005

2.12.1 Day 1

Problem 1. For $n = 1$ the rank is 1. Now assume $n \geq 2$. Since $A = (i)_{i,j=1}^n + (j)_{i,j=1}^n$, matrix A is the sum of two matrixes of rank 1. Therefore, the rank of A is at most 2. The determinant of the top-left 2×2 minor is -1, so the rank is exactly 2. Therefore, the rank of A is 1 for $n = 1$ and 2 for $n \geq 2$.

Solution 2. Consider the case $n \geq 2$. For $i = n, n-1, \dots, 2$, subtract the $(i-1)^{th}$ row from the n^{th} row. Then subtract the second row from all lower rows.

$$rank \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = rank \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} =$$

$$= rank \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 2.$$

Problem 2. Extend the definitions also for $n = 1, 2$. Consider the following sets

$$A'_n = \{(x_1, x_2, \dots, x_n) \in A_n : x_{n-1} = x_n\},\ A''_n = A_n \backslash A'_n,$$
$$B'_n = \{(x_1, x_2, \dots, x_n) \in B_n : x_n = 0\},\ B''_n = B_n \backslash B'_n$$

and denote $a_n = |A_n|, a'_n = |A'_n|, a''_n = |A''_n|, b_n = |B_n|, b'_n = |B'_n|, b''_n = |B''_n|$.

It is easy to observe the following relations between the a-sequences

$$\begin{cases} a_n & = a'_n + a''_n \\ a'_{n+1} & = a''_n \\ a''_{n+1} & = 2a'_n + 2a''_n \end{cases},$$

which lead to $a_{n+1} = 2a_n + 2a_{n-1}$.

For the b-sequences we have the same relations

$$\begin{cases} b_n & = b'_n + b''_n \\ b'_{n+1} & = b''_n \\ b''_{n+1} & = 2b'_n + 2b''_n \end{cases},$$

therefore $b_{n+1} = 2b_n + 2b_{n-1}$.

By computing the first values of (a_n) and (b_n) we obtain

$$\begin{cases} a_1 = 3,\ a_2 = 9,\ a_3 = 24 \\ b_1 = 3,\ b_2 = 8 \end{cases}$$

which leads to

$$\begin{cases} a_2 = 3b_1 \\ a_3 = 3b_2 \end{cases}$$

Now, reasoning by induction, it is easy to prove that $a_{n+1} = 3b_n$ for every $n \geq 1$.

Solution 2. Regarding x_i to be elements of $\mathbb{Z}_3$ and working "modulo 3" , we have that

$$(x_1, x_2, \ldots, x_n) \in A_n \Rightarrow (x_1 + 1, x_2 + 1, \ldots, x_n + 1) \in A_n,$$
$$(x_1 + 2, x_2 + 2, \ldots, x_n + 2 \in A_n$$

which means that $\frac{1}{3}$ of the elements of A_n start with 0. We establish a bijection between the subset of all the vectors in A_{n+1} which start with 0 and the set B_n by

$$(0, x_1, x_2, \ldots, x_n) \in A_{n+1} \mapsto (y_1, y_2, \ldots, y_n) \in B_n$$
$$y_1 = x_1, y_2 = x_2 - x_1, y_3 = x_3 - x_2, \ldots, y_n = x_n - x_{n-1}$$

(if $y_k = y_{k+1} = 0$ then $x_k - x_{k-1} = x_{k+1} - x_k = 0$ (where $x_0 = 0$), which gives $x_{k-1} = x_k = x_{k+1}$, which is not possible because of the definition of the sets A_p; therefore, the definition of the above function is correct).

The inverse is defined by

$$(y_1, y_2, \ldots, y_n) \in B_n \mapsto (0, x_1, x_2, \ldots, x_n) \in A_{n+1}$$
$$x_1 = y_1, x_2 = y_1 + y_2, \ldots, x_n = y_1 + y_2 + \cdots + y_n$$

Problem 3. Let $M = \max\limits_{0 \leq x \leq 1} |f'(x)|$. By the inequality $-M \leq f'(x) \leq M,\ x \in [0, 1]$ it follows:

$$-Mf(x) \leq f(x)f'(x) \leq Mf(x),\ x \in [0, 1].$$

By integration

$$-M \int_0^x f(t)dt \leq \frac{1}{2}f^2(x) - \frac{1}{2}f^2(0) \leq M \int_0^x f(t)dt,\ x \in [0, 1]$$

$$-Mf(x) \int_0^x f(t)dt \leq \frac{1}{2}f^3(x) - \frac{1}{2}f^2(0)f(x) \leq Mf(x) \int_0^x f(t)dt, x \in [0, 1].$$

Integrating the last inequality on $[0,1]$ it follows that

$$-M\Big(\int_0^1 f(x)dx\Big)^2 \le \int_0^1 f^3(x)dx - f^2(0)\int_0^1 f(x)dx \le M\Big(\int_0^1 f(x)dx\Big)^2 \Leftrightarrow$$

$$\Leftrightarrow \Big|\int_0^1 f^3(x)dx - f^2(0)\int_0^1 f(x)dx\Big| \le M\Big(\int_0^1 f(x)dx\Big)^2.$$

Solution 2. Let $M = \max\limits_{0\le x\le 1}|f'(x)|$ and $F(x) = -\int\limits_x^1 f$; then $F' = f, F(0) = -\int\limits_0^1 f$ and $F(1) = 0$. Integrating by parts,

$$\int_0^1 f^3 = \int_0^1 f^2.F' = [f^2F]_0^1 - \int_0^1 (f^2)'F =$$

$$= f^2(1)F(1) - f^2(0)F(0) - \int_0^1 2Fff' = f^2(0)\int_0^1 f - \int_0^1 2Fff'.$$

Then

$$\Big|\int_0^1 f^3(x)dx - f^2(0)\int_0^1 f(x)dx\Big| = \Big|\int_0^1 2Fff'\Big| \le \int_0^1 2Ff|f'| \le$$

$$\le M\int_0^1 2Ff = M.[F^2]_0^1 = M\Big(\int_0^1 f\Big)^2.$$

Problem 4. Note that $P(x)$ does not have any positive root because $P(x) > 0$ for every $x > 0$. Thus, we can represent them in the form $\alpha_i, i = 1, 2, \ldots, n$, where $\alpha_i \ge 0$. If $a_0 \ne 0$ then there is a $k \in \mathbb{N}, 1 \le k \le n-1$, with $a_k = 0$, so using Viete's formulae we get

$$\alpha_1\alpha_2\ldots\alpha_{n-k-1}\alpha_{n-k} + \alpha_1\alpha_2\ldots\alpha_{n-k-1}\alpha_{n-k+1} + \cdots + \alpha_{k+1}\alpha_{k+2}\ldots\alpha_{n-1}\alpha_n$$

$$= \frac{a_k}{a_n} = 0,$$

which is impossible because the left side of the equality is positive. Therefore $a_0 = 0$ and one of the roots of the polynomial, say α_n, must be equal to zero. Consider the polynomial $Q(x) = a_n x^{n-l} + a_{n-1}x^{n-2} + \cdots + a_1$. It has zeros $-\alpha_i, i = 1, 2, \ldots, n-1$. Again, Viete's formulae, for $n \geq 3$, yield:

$$\alpha_1\alpha_2\ldots\alpha_{n-1} = \frac{a_1}{a_n} \tag{1}$$

$$\alpha_1\alpha_2\ldots\alpha_{n-2} + \alpha_1\alpha_2\ldots\alpha_{n-3}\alpha_{n-1} + \cdots + \alpha_2\alpha_3\ldots\alpha_{n-1} = \frac{a_2}{a_n} \tag{2}$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \frac{a_{n-1}}{a_n}. \tag{3}$$

Dividing (2) by (1) we get

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_{n-1}} = \frac{a_2}{a_1}. \tag{4}$$

From (3) and (4), applying the AM-HM inequality we obtain

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}}{n-1} \geq \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_{n-1}}} = \frac{(n-1)a_1}{a_2},$$

therefore $\dfrac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$. Hence $\dfrac{n^2}{2} \geq \dfrac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$, implying $n \leq 3$. So, the only polynomials possibly satisfying (i) and (ii) are those of degree at most three. These polynomials can easily be found and they are $P(x) = x, P(x) = x^2 + 2x, P(x) = 2x^2 + x, P(x) = x^3 + 3x^2 + 2x$ and $P(x) = 2x^3 + 3x^2 + x$.

Solution 2. Consider the prime factorization of P in the ring $\mathbb{Z}[x]$. Since all roots of P are rational, P can be written as a product of n linear polynomials with rational coefficients. Therefore, all prime factor of P are linear and P can be written as

$$P(x) = \prod_{k=1}^{n}(b_k x + c_k)$$

where the coefficients b_k, c_k are integers. Since the leading coefficient of P is positive, we can assume $b_k > 0$ for all k. The coefficients of P are nonnegative, so P cannot have a positive root. This implies $c_k \geq 0$. It is not possible that $c_k = 0$ for two different values of k, because it would imply $a_0 = a_1 = 0$. So $c_k > 0$ in at least $n-1$ cases.

Now substitute $x = 1$.

$$P(1) = a_n + \cdots + a_0 = 0 + 1 + \cdots + n = \frac{n(n+1)}{2} = \prod_{k=1}^{n}(b_k + c_k) \geq 2^{n-1};$$

therefore it is necessary that $2^{n-1} \leq \frac{n(n+1)}{2}$, therefore $n \leq 4$. Moreover, the number $\frac{n(n+1)}{2}$ can be written as a product of $n-1$ integers greater than 1.

If $n = 1$, the only solution is $P(x) = 1x + 0$.

If $n = 2$, we have $P(1) = 3 = 1.3$, so one factor must be x, the other one is $x+2$ or $2x+1$. Both $x(x+2) = 1x^2 + 2x + 0$ and $x(2x+1) = 2x^2 + 1x + 0$ are solutions.

If $n = 3$, then $P(1) = 6 = 1.2.3$, so one factor must be x, another one is $x+1$, the third one is again $x+2$ or $2x+1$. The two polynomials are $x(x+1)(x+2) = 1x^3+3x^2+2x+0$ and $x(x+1)(2x+1) = 2x^3+3x^2+1x+0$, both have the proper set of coefficients.

In the case $n = 4$, there is no solution because $\frac{n(n+1)}{2}$ cannot be written as a product of 3 integers greater than 1.

Altogether we found 5 solutions: $1x+0, 1x^2+2x+0, 2x^2+1x+0, 1x^3+3x^2+2x+0$ and $2x^3+3x^2+1x+0$.

Problem 5. Let $g(x) = f'(x) + xf(x)$; then $f''(x) + 2xf'(x) + (x^2+1)f(x) = g'(x) + xg(x)$.

We prove that if h is a continuously differentiable function such that $h'(x) + xh(x)$ is bounded then $\lim\limits_{\infty} h = 0$. Applying this lemma for $h = g$ then for $h = f$, the statement follows.

Let M be an upper bound for $|h'(x)+xh(x)|$ and let $p(x) = h(x)e^{x^2/2}$. (The function $e^{-x^2/2}$ is a solution of the differential equation $u'(x) + xu(x) = 0$.) Then

$$|p'(x)| = |h'(x) + xh(x)|e^{x^2/2} \leq Me^{x^2/2}$$

and

$$|h(x)| = \left|\frac{p(x)}{e^{x^2/2}}\right| = \left|\frac{p(0) + \int\limits_0^x p'}{e^{x^2/2}}\right| \le \frac{|p(0)| + M\int\limits_0^x e^{x^2/2}dx}{e^{x^2/2}}.$$

Since $\lim\limits_{x\to\infty} e^{x^2/2} = \infty$ and $\lim \frac{\int\limits_0^x e^{x^2/2}dx}{e^{x^2/2}} = 0$ (by L'Hospital's rule), this implies $\lim\limits_{x\to\infty} h(x) = 0$.

Solution 2. Apply L'Hospital rule twice on the fraction $\frac{f(x)e^{x^2/x}}{e^{x^2/x}}$. (Note that L'Hospital rule is valid if the denominator converges to infinity, without any assumption on the numerator.)

$$\begin{aligned}\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^{x^2/2}}{e^{x^2/2}} &= \lim_{x\to\infty} \frac{(f'(x) + xf(x))e^{x^2/2}}{xe^{x^2/2}} \\ &= \lim_{x\to\infty} \frac{(f''(x) + 2xf'(x) + (x^2+1)f(x))e^{x^2/2}}{(x^2+1)e^{x^2/2}} = \\ &= \lim_{x\to\infty} \frac{f''(x) + 2xf'(x) + (x^2+1)f(x)}{x^2+1} = 0.\end{aligned}$$

Problem 6. Write $d = gcd(m, n)$. It is easy to see that $< G(m), G(n) >= G(d)$; hence, it will suffice to check commutativity for any two elements in $G(m) \cup G(n)$, and so for any two generators a^m and b^n. Consider their commutator $z = a^{-m}b^{-n}a^m b^n$; then the relations

$$z = (a^{-m}ba^m)^{-n}b^n = a^{-m}(b^{-n}ab^n)^m$$

show that $z \in G(m) \cap G(n)$. But then z is in the center of $G(d)$. Now, from the relation $a^m b^n = b^n a^m z$, it easily follows by induction that

$$a^{ml}b^{nl} = b^{nl}a^{ml}z^{l^2}.$$

Setting $l = \frac{m}{d}$ and $l = \frac{n}{d}$ we obtain $z^{(m/d)^2} = z^{(n/d)^2} = e$, but this implies that $z = e$ as well.

2.12.2 Day 2

Problem 1. Write $f(x) = (x + \frac{b}{2})^2 + d$ where $d = c - \frac{b^2}{4}$. The absolute minimum of f is d.

If $d \geq 1$ then $f(x) \geq 1$ for all $x, M = \emptyset$ and $|M| = 0$.

If $-1 < d < 1$ then $f(x) > -1$ for all x,

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \Leftrightarrow \left|x + \frac{b}{2}\right| < \sqrt{1-d}$$

so

$$M = \Big(-\frac{b}{2} - \sqrt{1-d}, -\frac{b}{2} + \sqrt{1-d}\Big)$$

and

$$|M| = 2\sqrt{1-d} < 2\sqrt{2}.$$

If $d \leq -1$ then

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \Leftrightarrow \sqrt{|d|-1} < \left|x + \frac{b}{2}\right| < \sqrt{|d|+1}$$

so

$$M = (-\sqrt{|d|+1}, -\sqrt{|d|-1}) \cup (\sqrt{|d|-1}, \sqrt{|d|+1})$$

and

$$|M| = 2(\sqrt{|d|+1} - \sqrt{|d|-1}) = 2\frac{(|d|+1) - (|d|-1)}{\sqrt{|d|+1} + \sqrt{|d|-1}} \leq$$
$$\leq 2\frac{2}{\sqrt{1+1} + \sqrt{1-0}} = 2\sqrt{2}.$$

Problem 2. Yes, it is even enough to assume that f^2 and f^3 are polynomials.

Let $p = f^2$ and $q = f^3$. Write these polynomials in the form of

$$p = a.p_2^{a_1} \dots p_k^{a_k}, \ \ q = b.q_1^{b_1} \dots q_l^{b_l},$$

where $a, b \in \mathbb{R}$, $a_1, \dots, a_k$, $b_1, \dots, b_l$ are positive integers and $p_1, \dots, p_k$, $q_1, \dots, q_l$ are irreducible polynomials with leading coefficients 1. For $p^3 = q^2$ and the factorisation of $p^3 = q^2$ is unique we get that $a^3 = b^2, k = 1$ and for some $(i_1, \dots, i_k)$ permutation of $(1, \dots, k)$ we have $p_1 = q_{i_1}, \dots, p_k = q_{i_k}$ and $3a_1 = 2b_{i_1}, \dots, 3a_k = 2b_{i_k}$. Hence $b_1, \dots, b_l$ are divisible by 3 let $r = b^{1/3}.q_1^{b_1/3} \dots q_l^{b_l/3}$ be a polynomial. Since $r^3 = q = f^3$ we have $f = r$.

Solution 2. Let $\frac{p}{q}$ be the simplest form of the rational function $\frac{f^3}{f^2}$. Then the simplest form of its square is $\frac{p^2}{q^2}$. On the other hand $\frac{p^2}{q^2} = \left(\frac{f^3}{f^2}\right)^2 = f^2$ is a polynomial therefore q must be a constant and so $f = \frac{f^3}{f^2} = \frac{p}{q}$ is a polynomial.

Problem 3. If A is a nonzero symmetric matrix, then $trace(A^2) = trace(A^tA)$ is the sum of the squared entries of A which is positive. So V cannot contain any symmetric matrix but 0.

Denote by S the linear space of all real $n \times n$ symmetric matrices; $dimV = \frac{n(n+1)}{2}$. Since $V \cap S = \{0\}$, we have $dimV + dimS \leq n^2$ and thus $dimV \leq n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the condition of the problem.

Therefore the maximum dimension of V is $\frac{n(n-1)}{2}$.

Problem 4. Let

$$g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that $g(\pm 1) = f(\pm 1), g(0) = f(0)$ and $g'(0) = f'(0)$.

Apply Rolle's theorem for the function $h(x) = f(x) - g(x)$ and its derivatives. Since $h(-1) = h(0) = h(1) = 0$, there exist $\eta \in (-1, 0)$ and $\vartheta \in (0, 1)$ such that $h'(\eta) = h'(\vartheta) = 0$. We also have $h'(0) = 0$, so there exist $\varrho \in (\eta, 0)$ and $\sigma \in (0, \vartheta)$ such that $h''(\varrho) = h''(\sigma) = 0$. Finally, there exists a $\xi \in (\varrho, \sigma) \subset (-1, 1)$ where $h'''(\xi) = 0$. Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2}.6 - f(0).0 + \frac{f(1)}{2}.6 - f'(0).6 = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 2. The expression $\frac{f(1) - f(-1)}{2} - f'(0)$ is the divided difference $f[-l, 0, 0, 1]$ and there exists a number $\xi \in (-1, 1)$ such that $f[-1, 0, 0, 1] = \frac{f'''(\xi)}{3!}$.

Problem 5. To get an upper bound for r, set $f(x,y) = x - \frac{x^2}{2} + \frac{y^2}{2}$. This function satisfies the conditions, since $\operatorname{grad} f(x,y) = (1-x, y)$, $\operatorname{grad} f(0,0) = (1,0)$ and $(|\operatorname{grad} f(x_1,y_1) - \operatorname{grad} f(x_2,y_2)| = |(x_2 - x_1, y_1 - y_2)| = |(x_1,y_1) - (x_2,y_2)|$

In the disk $D_r = \{(x,y) : x^2 + y^2 \leq r^2\}$

$$f(x,y) = \frac{x^2+y^2}{2} - \Big(x - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{r^2}{2} + \frac{1}{4}.$$

If $r > \frac{1}{2}$ then the absolute maximum is $\frac{r^2}{2} + \frac{1}{4}$, attained at the points $\left(\frac{1}{2}, \pm\sqrt{r^2 - \frac{1}{4}}\right)$. Therefore, it is necessary that $r \leq \frac{1}{2}$ because if $r > \frac{1}{2}$ then the maximum is attained twice.

Suppose now that $r \leq \frac{1}{2}$ and that f attains its maximum on D_r at $u, v, u \neq v$. Since $|\operatorname{grad} f(z) - \operatorname{grad} f(0)| \leq r, |\operatorname{grad} f(z)1 \leq 1 - r > 0$ for all $z \in D_r$. Hence f may attain its maximum only at the boundary of D_r, so we must have $|u| = |v| = r$ and $\operatorname{grad} f(u) = au$ and $\operatorname{grad} f(v) = bv$, where $a, b \geq 0$. Since $au = \operatorname{grad} f(u)$ and $bv = \operatorname{grad} f(v)$ belong to the disk D with centre $\operatorname{grad} f(0)$ and radius r, they do not belong to the interior of D_r. Hence $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| = |au - bv| \geq |u - v|$ and this inequality is strict since $D \cap D_r$ contains no more than one point. But this contradicts the assumption that $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| \leq |u - v|$. So all $r \leq \frac{1}{2}$ satisfies the condition.

Problem 6. First consider the case when $q = 0$ and r is rational. Choose a positive integer t such that $r^2 t$ is an integer and set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + rt & -r^2 t \\ t & 1 - rt \end{pmatrix}.$$

Then

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \frac{ar + b}{cr + d} = \frac{(1+rt)r - r^2 t}{tr + (1 - rt)} = r.$$

Now assume $q \neq 0$. Let the minimal polynomial of r in $\mathbb{Z}[x]$ be $ux^2 + vx + w$. The other root of this polynomial is $\overline{r} = p - q\sqrt{7}$, so $v =$

$-u(r+\overline{r}) = -2up$ and $w = ur\overline{r} = u(p^2 - 7q^2)$. The discriminant is $v^2 - 4uw = 7.(2uq)^2$. The left-hand side is an integer, implying that also $\triangle = 2uq$ is an integer.

The equation $\dfrac{ar+b}{cr+d} = r$ is equivalent to $cr^2 + (d-a)r - b = 0$. This must be a multiple of the minimal polynomial, so we need

$$c = ut, \; d - a = vt, \; -b = wt$$

for some integer $t \neq 0$. Putting together these equalities with $ad - bc = 1$ we obtain that

$$(a+d)^2 = (a-d)^2 + 4ad = 4 + (v^2 - 4uw)t^2 = 4 + 7\triangle^2 t^2.$$

Therefore $4 + 7\triangle^2 t^2$ must be a perfect square. Introducing $s = a + d$, we need an integer solution (s,t) for the Diophantine equation

$$s^2 - 7\triangle^2 t^2 = 4 \tag{1}$$

such that $t \neq 0$.

The numbers s and t will be even. Then $a + d = s$ and $d - a = vt$ will be even as well and a and d will be really integers.

Let $(8 \pm 3\sqrt{7})^n = k_n \pm l_n\sqrt{7}$ for each integer n. Then $k_n^2 - 7l_n^2 = (k_n + l_n\sqrt{7})(k_n - l_n\sqrt{7}) = ((8+3\sqrt{7})^n(8-3\sqrt{7}))^n = 1$ and the sequence (l_n) also satisfies the linear recurrence $l_{n+1} = 16l_n - l_{n-1}$. Consider the residue of l_n modulo $\triangle$. There are $\triangle^2$ possible residue pairs for (l_n, l_{n+1}) so some are the same. Starting from such two positions, the recurrence shows that the sequence of residues is periodic in both directions. Then there are infinitely many indices such that $l_n \equiv l_0 = 0 \pmod{\triangle}$.

Taking such an index n, we can set $s = 2k_n$ and $t = 2l_n/\triangle$.

Remarks. 1. It is well-known that if $D > 0$ is not a perfect square then the Pell-like Diophantine equation

$$x^2 + Dy^2 = 1$$

has infinitely many solutions. Using this fact the solution can be generalized to all quadratic algebraic numbers.

2. It is also known that the continued fraction of a real number r is periodic from a certain point if and only if r is a root of a quadratic equation. This fact can lead to another solution.

2.13 Solutions of Olympic 2006

2.13.1 Day 1

Problem 1.

a) False. Consider function $f(x) = x^3 - x$. It is continuous, $range(f) = \mathbb{R}$ but, for example, $f(0) = 0, f(\frac{1}{2}) = -\frac{3}{8}$ and $f(1) = 0$, therefore $f(0) > f(\frac{1}{2})$, $f(\frac{1}{2}) < f(1)$ and f is not monotonic.

b) True. Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim_{a-} f$ and $\lim_{a+} f$ exist and $\lim_{a-} f \leq \lim_{a+} f$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim_{a-} f = b < \lim_{a+} f = c$, we have $f(x) \leq b$ for all $x < a$ and $f(x) \geq c$ for all $x > a$; therefore $range(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$ cannot be the complete $\mathbb{R}$.

For non-increasing f the same can be applied writing reverse relations or $g(x) = -f(x)$.

c) False. The function $g(x) = \arctan x$ is monotonic and continuous, but $range(g) = (-\frac{\pi}{2}, \frac{\pi}{2}) \neq \mathbb{R}$.

Problem 2. Let $S_k = \{0 < x < 10^k | x^2 - x \text{ is divisible by } 10^k\}$ and $s(k) = |S_k|, k \geq 1$. Let $x = a_{k+1}a_k \ldots a_1$ be the decimal writing of an integer $x \in S_{k+1}, k \geq 1$. Then obviously $y = a_k \ldots a_1 \in S_k$. Now, let $y = a_k \ldots a_1 \in S_k$ be fixed. Considering a_{k+1} as a variable digit, we have $x^2 - x = (a_{k+1}10^k + y)^2 - (a_{k+1}10^k + y) = (y^2 - y) + a_{k+1}10^k(2y - 1) + a_{k+1}^2 10^{2k}$. Since $y^2 - y = 10^k z$ for an iteger z, it follows that $x^2 - x$ is divisible by 10^{k+1} if and only if $z + a_{k+1}(2y - 1) \equiv 0 \pmod{10}$. Since $y \equiv 3 \pmod{10}$ is obviously impossible, the congruence has exactly one solution. Hence we obtain a one-to-one correspondence between the sets S_{k+1} and S_k for every $k \geq$. Therefore $s(2006) = s(1) = 3$, because

$S_1 = \{1, 5, 6\}$.

Solution 2. Since $x^2 - x = x(x-1)$ and the numbers x and $x-1$ are relatively prime, one of them must be divisible by 2^{2006} and one of them (may be the same) must be divisible by 5^{2006}. Therefore, x must satisfy the following two conditions:

$$x \equiv 0 \text{ or } 1 \pmod{2^{2006}};$$
$$x \equiv 0 \text{ or } 1 \pmod{5^{2006}}.$$

Altogether we have 4 cases. The Chinese remainder theorem yields that in each case there is a unique solution among the numbers $0, 1, \ldots, 10^{2006} - 1$. These four numbers are different because each two gives different residues modulo 2^{2006} or 5^{2006}. Moreover, one of the numbers is 0 which is not allowed.

Therefore there exist 3 solutions.

Problem 3. By induction, it is enough to consider the case $m = 2$. Furthermore, we can multiply A with any integral matrix with determinant 1 from the right or from the left, without changing the problem. Hence we can assume A to be upper triangular.

Lemma. Let A be an integral upper triangular matrix, and let b, c be integers satisfying $detA = bc$. Then there exist integral upper triangular matrices B, C such that $detB = b, detC = c, A = BC$.

Proof. The proof is done by induction on n, the case $n = 1$ being obvious. Assume the statement is true for $n-1$. Let A, b, c as in the statement of the lemma. Define B_{nn} to be the greatest common divisor of b and A_{nn}, and put $C_{nn} = \dfrac{A_{nn}}{B_{nn}}$. Since A_{nn} divides bc, C_{nn} divides $\dfrac{b}{B_{nn}}c$, which divides c. Hence C_{nn} divides c. Therefore, $b' = \dfrac{b}{B_{nn}}$ and $c' = \dfrac{c}{C_{nn}}$ are integers. Define A' to be the upper-left $(n-1) \times (n-1)$-submatrix of A; then $detA' = b'c'$. By induction we can find the upper-left $(n-1) \times (n-1)$-part of B and C in such a way that $detB = b, detC = c$ and $A = BC$ holds on the upper-left $(n-1) \times (n-1)$-submatrix of A.

It remains to define $B_{i,n}$ and $C_{i,n}$ such that $A = BC$ also holds for the (i,n)-th entry for all $i < n$.

First we check that B_{ii} and C_{nn} are relatively prime for all $i < n$. Since B_{ii} divides b', it is certainly enough to prove that b' and C_{nn} are relatively prime, i.e.

$$gcd\Big(\frac{b}{gcd(b,A_{nn})}, \frac{A_{nn}}{gcd(b,A_{nn})}\Big) = 1,$$

which is obvious. Now we define $B_{j,n}$ and $C_{j,n}$ inductively: Suppose we have defined $B_{i,n}$ and $C_{i,n}$ for all $i = j+1, j+2, \ldots, n-1$. Then $B_{j,n}$ and $C_{j,n}$ have to satisfy

$$A_{j,n} = B_{j,j}C_{j,n} + B_{j,j+1}C_{j+1,n} + \cdots + B_{j,n}C_{n,n}$$

Since $B_{j,j}$ and $C_{n,n}$ are relatively prime, we can choose integers $C_{j,n}$ and $B_{j,n}$ such that this equation is satisfied. Doing this step by step for all $j = n-1, n-2, \ldots, 1$, we finally get B and C such that $A = BC$.

Problem 4. Let S be an infinite set of integers such that rational function $f(x)$ is integral for all $X \in S$. Suppose that $f(x) = \dfrac{p(x)}{q(x)}$ where p is a polynomial of degree k and q is a polynomial of degree n. Then p, q are solutions to the simultaneous equations $p(x) = q(x)f(x)$ for all $x \in S$ that are not roots of q. These are linear simultaneous equations in the coefficients of p, q with rational coefficients. Since they have a solution, they have a rational solution.

Thus there are polynomials p', q' with rational coefficients such that $p'(x) = q'(x)f(x)$ for all $x \in S$ that are not roots of q. Multiplying this with the previous equation, we see that $p'(x)q(x)f(x) = p(x)q'(x)f(x)$ for all $x \in S$ that are not roots of q. If x is not a root of p or q, then $f(x) \neq 0$, and hence $p'(x)q(x) = p(x)q'(x)$ for all $x \in S$ except for finitely many roots of p and q. Thus the two polynomials $p'q$ and pq' are equal for infinitely many choices of value. Thus $p'(x)q(x) = p(x)q'(x)$. Dividing by $q(x)q'(x)$, we see that $dfrac p'(x)q'(x) = \dfrac{p(x)}{q(x)} = f(x)$. Thus

$f(x)$ can be written as the quotient of two polynomials with rational coefficients. Multiplying up by some integer, it can be written as the quotient of two polynomials with integer coefficients.

Suppose $f(x) = \dfrac{p''(x)}{q''(x)}$ where p'' and q'' both have integer coefficients. Then by Euler's division algorithm for polynomials, there exist polynomials s and r, both of which have rational coefficients such that $p''(x) = q''(x)s(x) + r(x)$ and the degree of r is less than the degree of q''. Dividing by $q''(x)$, we get that $f(x) = s(x) + \dfrac{r(x)}{q''(x)}$. Now there exists an integer N such that $Ns(x)$ has integral coefficients. Then $Nf(x) - Ns(x)$ is an integer for all $x \in S$. However, this is equal to the rational function $\dfrac{Nr}{q''}$, which has a higher degree denominator than numerator, so tends to 0 as x tends to ∞. Thus for all sufficiently large $x \in S, Nf(x) - Ns(x) = 0$ and hence $r(x) = 0$. Thus r has infinitely many roots, and is 0. Thus $f(x) = s(x)$, so f is a polynomial.

Problem 5. Without loss of generality $a \geq b \geq c, d \geq e$. Let $c^2 = e^2 + \triangle, \triangle \in \mathbb{R}$. Then $d^2 = a^2 + b^2 + \triangle$ and the second equation implies

$$a^4 + b^4 + (e^2 + \triangle)^2 = (a^2 + b^2 + \triangle)^2 + e^4,\ \triangle = -\frac{a^2b^2}{a^2 + b^2 - e^2}.$$

(Here $a^2 + b^2 - e^2) \geq \frac{2}{3}(a^2 + b^2 + c^2) - \frac{1}{2}(d^2 + e^2) = \frac{1}{6}(d^2 + e^2) > 0)$.

Since $c^2 = e^2 - \dfrac{a^2b^2}{a^2 + b^2 - e^2} = \dfrac{(a^2 - e^2)(e^2 - b^2)}{a^2 + b^2 - e^2} > 0$ then $a > e > b$.

Therefore $d^2 = a^2 + b^2 - \dfrac{a^2b^2}{a^2 + b^2 - e^2} < a^2$ and $a > d \geq e > b \geq c$.

Consider a function $f(x) = a^x + b^x + c^x - d^x - e^x,\ x \in \mathbb{R}$. We shall prove that $f(x)$ has only two zeroes $x = 2$ and $x = 4$ and changes the sign at these points. Suppose the contrary. Then Rolle's theorem implies that $f'(x)$ has at least two distinct zeroes. Without loss of generality $a = 1$. Then $f'(x) = \ln b.b^x + \ln c.c^x - \ln d.d^x - \ln e.e^x, x \in \mathbb{R}$. If $f'(x_1) = f'(x_2) = 0, x_1 < x_2$, then $\ln b.b^{x_i} + \ln c.c^{x_i} = \ln d.d^{x_i} + \ln e.e^{x_i}, i = 1, 2,$

but since $1 > d \geq e > b \geq c$ we have

$$\frac{(-\ln b).b^{x_2} + (-\ln c).c^{x_2}}{(-\ln b).b^{x^1} + (-\ln c).c^{x_1}} \leq b^{x_2 - x_1} < e^{x_2 - x_1} \leq \frac{(-\ln d).d^{x_2} + (-\ln e).e^{x_2}}{(-\ln d).d^{x^1} + (-\ln e).e^{x_1}},$$

a contradiction. Therefore $f(x)$ has a constant sign at each of the intervals $(-\infty, 2), (2, 4)$ and $(4, \infty)$. Since $f(0) = 1$ then $f(x) > 0, x \in (-\infty, 2) \cup (4, \infty)$ and $f(x) < 0, x \in (2, 4)$. In particular, $f(3) = a^3 + b^3 + c^3 - d^3 - e^3 < 0$.

Problem 6. Let $A(x) = a_0 + a_1x + \cdots + a_nx^n$. We prove that sequence $a_0, \ldots, a_n$ satisfies the required property if and only if all zeros of polynomial $A(x)$ are real.

a) Assume that all roots of $A(x)$ are real. Let us use the following notations. Let I be the identity operator on $\mathbb{R} \to \mathbb{R}$ functions and D be differentiation operator. For an arbitrary polynomial $P(x) = p_0 + p_1x + \cdots + p_nx^n$, write $P(D) = p_0I + p_1D + p_2D^2 + \cdots + p_nD^n$. Then the statement can written as $(A(D)f)(\xi) = 0$.

First prove the statement for $n = 1$. Consider the function

$$g(x) = e^{\frac{a_0}{a_1}} f(x).$$

Since $g(x_0) = g(x_1) = 0$, by Rolle's theorem there exists $a\xi \in (x_0, x_1)$ for which

$$g'(\xi) = \frac{a_0}{a_1} e^{\frac{a_0}{a_1}\xi} f(\xi) + e^{\frac{a_0}{a_1}\xi} f'(\xi) = \frac{e^{\frac{a_0}{a_1}\xi}}{a_1}(a_0f(\xi) + a_1f'(\xi)) = 0.$$

Now assume that $n > 1$ and the statement holds for $n - 1$. Let $A(x) = (x - c)B(x)$ where c is a real root of polynomial A. By the $n = 1$ case, there exist $y_0 \in (x_0, x_1), y_1 \in (x_1, x_2), \ldots, y_{n-1} \in (x_{n-1}, x_n)$ such that $f'(y_j) - cf(y_j) = 0$ for all $j = 0, 1, \ldots, n - 1$. Now apply the induction hypothesis for polynomial $B(x)$, function $g = f' - cf$ and points $y_0, \ldots, y_{n-1}$. The hypothesis says that there exists $a \in (y_0, y_{n-1} \subset (x_0, x_n)$ such that

$$(B(D)g)(\xi) = (B(D)(D - cI)f)(\xi) = (A(D)f)(\xi) = 0.$$

b) Assume that $u+vi$ is a complex root of polynomial $A(x)$ such that $v \neq 0$. Consider the linear differential equation $a_n g^{(n)} + \cdots + a_1 g' + g = 0$. A solution of this equation is $g_1(x) = e^{ux} \sin vx$ which has infinitely many zeros.

Let k be the smallest index for which $a_k \neq 0$. Choose a small $\epsilon > 0$ and set $f(x) = g_1(x) + \epsilon x^k$. If ϵ is sufficiently small then g has the required number of roots but $a_0 f + a_1 f' + \cdots + a_n f^{(n)} = a_k \epsilon \neq 0$ everywhere.

2.13.2 Day 2

Problem 1. Apply induction on n. For the initial cases $n = 3, 4, 5$, chose the triangulations shown in the Figure to prove the statement.

Now assume that the statement is true for some $n = k$ and consider the case $n = k + 3$. Denote the vertices of V by $P_1, \ldots, P_{k+3}$ Apply the induction hypothesis on the polygon $P_1 P_2 \ldots P_k$ in this triangulation each of vertices $P_1, \ldots, P_k$ belong to an odd number of triangles, except two vertices if n is not divisible by 3. Now add triangles $P_1 P_k P_{k+2}, P_k P_{k+1} P_{k+2}$ and $P_1 P_{k+2} P_{k+3}$. This way we introduce two new triangles at vertices P_1 and P_k so parity is preserved. The vertices P_{k+l}, P_{k+2} and P_{k+3} share an odd number of triangles. Therefore, the number of vertices shared by even number of triangles remains the same as in polygon $P_1 P_2 \ldots P_k$.

Problem 2. The functions $f(x) = x + c$ and $f(x) = -x + c$ with some constant c obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let f be such a function. Then f clearly satisfies $|f(x) - f(y)| \leq |x - y|$ for all x, y; therefore, f is continuous. Given x, y with $x < y$, let $a, b \in [x, y]$ be such that $f(a)$ is the maximum and $f(b)$ is the minimum of f on $[x, y]$. Then $f([x, y]) = [f(b), f(a)]$; hence

$$y - x = f(a) - f(b) \leq |a - b| \leq y - x$$

This implies $\{a, b\} = \{x, y\}$, and therefore f is a monotone function.

Suppose f is increasing. Then $f(x) - f(y) = x - y$ implies $f(x) - x = f(y) - y$, which says that $f(x) = x + c$ for some constant c. Similarly, the case of a decreasing function f leads to $f(x) = -x + c$ for some constant c.

Problem 3. Let $f(x) = \tan(\sin x) - \sin(\tan x)$. Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x).\cos^2(\sin x)}{\cos^2(x).\cos^2(\tan x)}$$

Let $0 < x < \arctan\frac{\pi}{2}$. It follows from the concavity of cosine on $(0, \frac{\pi}{2})$ that

$$\begin{aligned}\sqrt[3]{\cos(\tan x).\cos^2(\sin x)} &< \frac{1}{3}[\cos(\tan x) + 2\cos(\sin x)] \\ &\leq \cos\left[\frac{\tan x + 2\sin x}{3}\right] < \cos x,\end{aligned}$$

the last inequality follows from

$$\left[\frac{\tan x + 2\sin x}{3}\right]' = \frac{1}{3}\left[\frac{1}{\cos^2 x} + 2\cos x\right] \geq \sqrt[3]{\frac{1}{\cos^2 x}.\cos x.\cos x} = 1.$$

This proves that $\cos^3 x - \cos(\tan x).\cos^2(\sin x) > 0$, so $f'(x) > 0$, so f increases on the interval $[0, \arctan\frac{\pi}{2}]$. To end the proof it is enough to notice that (recall that $4 + \pi^2 < 16$)

$$\tan\left[\sin\left(\arctan\frac{\pi}{2}\right)\right] = \tan\frac{\pi/2}{\sqrt{1 + \pi^2/4}} > \tan\frac{\pi}{4} = 1.$$

This implies that if $x \in [\arctan\frac{\pi}{2}, \frac{\pi}{2}]$ then $\tan(\sin x) > 1$ and therefore $f(x) > 0$.

Problem 4. By passing to a subspace we can assume that $v_1, \ldots, v_n$ are linearly independent over the reals. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ satisfying

$$v_{n+1} = \sum_{j=1}^{n} \lambda_j v_j$$

We shall prove that λ_j is rational for all j. From

$$-2 < v_i, v_j >= |v_i - v_j|^2 - |v_i|^2 - |v_j|^2$$

we get that $< v_i, v_j >$ is rational for all i, j. Define A to be the rational $n \times n$-matrix $A_{ij} =< v_i, v_j >, w \in \mathbb{Q}^n$ to be the vector $w_i =< v_i, v_{n+1} >$, and $\lambda \in \mathbb{R}^n$ to be the vector $(\lambda_i)_i$ Then,

$$< v_i, v_{i+1} >= \sum_{j=1}^{n} \lambda_j < v_i, v_j >$$

gives $A\lambda = w$. Since $v_1, \ldots, v_n$ are linearly independent, A is invertible. The entries of A^{-1} are rationals, therefore $\lambda = A^{-1}w \in \mathbb{Q}^n$, and we are done.

Problem 5. Substituting $y = x + m$, we can replace the equation by

$$y^3 - ny + mn = 0.$$

Let two roots be u and v; the third one must be $w = -(u + v)$ since the sum is 0. The roots must also satisfy

$$uv + uw + vw = -(u^2 + uv + v^2) = -n, \ i.e. \ u^2 + uv + v^2 = n$$

and

$$uvw = -uv(u + v) = mn.$$

So we need some integer pairs (u, v) such that $uv(u + v)$ is divisible by $u^2 + uv + v^2$. Look for such pairs in the form $u = kp, v = kq$. Then

$$u^2 + uv + v^2 = k^2(p^2 + pq + q^2),$$

and

$$uv(u + v) = k^3pq(p + q).$$

Chosing p, q such that they are coprime then setting $k = p^2 + pq + q^2$ we have $\dfrac{uv(u + v)}{u^2 + uv + v^2} = p^2 + pq + q^2$.

Substituting back to the original quantites, we obtain the family of cases

$$m = (p^2 + pq + q^2)^3, \ \ m = p^2q + pq^2,$$

and the three roots are

$$x_1 = p^3, \ x_2 = q^3, \ x_3 = -(p + q)^3.$$

Problem 6. We note that the problem is trivial if $A_j = \lambda I$ for some j, so suppose this is not the case. Consider then first the situation where some A_j, say A_3, has two distinct real eigenvalues. We may assume that $A_3 = B_3 = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$ by conjugating both sides. Let $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then

$$a + d = TrA_2 = TrB_2 = a' + d'$$
$$a\lambda + d\mu = Tr(A_2A_3) = Tr(A_1^{-1} = TrB_1^{-1} = Tr(B_2B_3) = a'\lambda + d'\mu.$$

Hence $a = a'$ and $d = d'$ and so also $bc = b'c'$. Now we cannot have $c = 0$ or $b = 0$, for then $(1, 0)^T$ or $(0, 1)^T$ would be a common eigenvector of all A_j. The matrix $S = \begin{pmatrix} c' & \\ & c \end{pmatrix}$ conjugates $A_2 = S^{-1}B_2S$, and as S commutes with $A_3 = B_3$, it follows that $A_j = S^{-1}B_jS$ for all j.

If the distinct eigenvalues of $A_3 = B_3$ are not real, we know from above that $A_j = S^{-1}B_jS$ for some $S \in GL_2\mathbb{C}$ unless all A_j have a common eigenvector over $\mathbb{C}$. Even if they do, say $A_jv = \lambda_jv$, by taking the conjugate square root it follows that $A_j's$ can be simultaneously diagonalized. If $A_2 = \begin{pmatrix} a & \\ & d \end{pmatrix}$ and $B_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, it follows as above that $a = a', d = d'$ and so $b'c' = 0$. Now B_2 and B_3 (and hence B_1 too) have a common eigenvector over $\mathbb{C}$ so they too can be simultaneously diagonalized. And so $SA_j = B_jS$ for some $S \in GL_2\mathbb{C}$ in either case. Let $S_0 = ReS$ and $S_1 = ImS$. By separating the real and imaginary components, we are done if either S_0 or S_1 is invertible. If not, So may be conjugated to some $T^{-1}S_0T = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$, with $(x, y)^T \neq (0, 0)^T$, and it follows that all A_j have a common eigenvector $T(0, 1)^T$, a contradiction.

We are left with the case when n_0A_j has distinct eigenvalues; then these eigenvalues by necessity are real. By conjugation and division by scalars we may assume that $A_3 = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ and $b \neq 0$. By further conjugation by upper-triangular matrices (which preserves the shape of A_3 up to the value of b) we can also assume that $A_2 = \begin{pmatrix} 0 & u \\ 1 & v \end{pmatrix}$. Here

$v^2 = Tr^2A_2 = 4detA_2 = -4u$. Now $A_1 = A_3^{-1}A_2^{-1} = \begin{pmatrix} -(b+v)/u & 1 \\ 1/u & \end{pmatrix}$, and hence $\frac{b+v)^2}{u^2} = Tr^2A_1 = 4detA_1 = -\frac{4}{u}$. Comparing these two it follows that $b = -2v$. What we have done is simultaneously reduced all A_j to matrices whose all entries depend on u and v ($= -detA_2$ and TrA_2, respectively) only, but these themselves are invariant under similarity. So B_j's can be simultaneously reduced to the very same matrices.

2.14 Solutions of Olympic 2007

2.14.1 Day 1

Problem 1. Let $f(x) = ax^2 + bx + c$. Substituting $x = 0, x = 1$ and $x = -1$, we obtain that $5|f(0) = c, 5|f(1) = (a + b + c)$ and $5|f(-1) = (a - b + c)$. Then $5|f(1) + f(-1) - 2f(0) = 2a$ and $5|f(1) - f(-1) = 2b$. Therefore 5 divides $2a, 2b$ and c and the statement follows.

Solution 2. Consider $f(x)$ as a polynomial over the 5-element field (i.e. modulo 5). The polynomial has 5 roots while its degree is at most 2. Therefore $f \equiv 0 \pmod 5$ and all of its coefficients are divisible by 5.

Problem 2. The minimal rank is 2 and the maximal rank is n. To prove this, we have to show that the rank can be 2 and n but it cannot be 1.

(i) The rank is at least 2. Consider an arbitrary matrix $A = [a_{ij}]$ with entries $1, 2, \ldots, n^2$ in some order. Since permuting rows or columns of a matrix does not change its rank, we can assume that $1 = a_{11} < a_{21} < \cdots < a_{n1}$ and $1 = a_{11} < a_{12} < \cdots < a_{1n}$. Hence $a_{n1} \geqslant n$ and $a_{1n} \geqslant n$ and at least one of these inequalities is strict. Then $\det \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} < 1.n^2 - n.n = 0$ so $\mathrm{rk}(A) \geqslant \mathrm{rk} \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} \geqslant 2$.

(ii) The rank can be 2. Let

$$T = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix}$$

The i-th row is $(1,2,\dots,n) + n(i-1).(1,1,\dots,1)$ so each row is in the two-dimensional subspace generated by the vectors $(1,2,\dots,n)$ and $(1,1,\dots,1)$. We already proved that the rank is at least 2, so $\mathrm{rk}(T) = 2$.

(iii) The rank can be n, i.e. the matrix can be nonsingular. Put odd numbers into the diagonal, only even numbers above the diagonal and arrange the entries under the diagonal arbitrarily. Then the determinant of the matrix is odd, so the rank is complete.

Problem 3. The possible values for k are 1 and 2.

If $k = 1$ then $P(x) = \alpha x^2$ and we can choose $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$.

If $k = 2$ then $P(x,y) = \alpha x^2 + \beta y^2 + \gamma xy$ and we can choose matrices $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & \beta \\ -1 & \gamma \end{pmatrix}$.

Now let $k \geqslant 3$. We show that the polynomial $P(x_1,\dots,x_k) = \sum\limits_{i=0}^{k} x_i^2$ is not good. Suppose that $P(x_1,\dots,x_k) = \det\left(\sum\limits_{i=0}^{k} x_i A_i\right)$. Since the first columns of $A_1,\dots,A_k$ are linearly dependent, the first column of some non-trivial linear combination $y_1A_1 + \cdots + y_kA_k$ is zero. Then $\det(y_1A_1 + \cdots + y_kA_k) = 0$ but $P(y_1,\dots,y_k) \neq 0$, a contradiction.

Problem 4. We start with three preliminary observations.

Let U, V be two arbitrary subsets of G. For each $x \in U$ and $y \in V$ there is a unique $z \in G$ for which $xyz = e$. Therefore,

$$N_{UVG} = |U \times V| = |U|.|V|.$$

Second, the equation $xyz = e$ is equivalent to $yzx = e$ and $zxy = e$. For arbitrary sets $U, V, W \subset G$, this implies

$$\begin{aligned} \{(x,y,z) \in U \times V \times W : xyz = e\} &= \{(x,y,z) \in U \times V \times W : yzx = e\} \\ &= \{(x,y,z) \in U \times V \times W : zxy = e\} \end{aligned}$$

and therefore

$$N_{UVW} = N_{VWU} = N_{WUV}.$$

Third, if $U, V \subset G$ and W_1, W_2, W_3 are disjoint sets and $W = W_1 \cup W_2 \cup W_3$ then, for arbitrary $U, V \subset G$,

$$\{(x,y,z) \in U \times V \times W : xyz = e\} = \{(x,y,z) \in U \times V \times W_1 : xyz = e\} \cup$$
$$\cup \{(x,y,z) \in U \times V \times W_2 : xyz = e\} \cup \{(x,y,z) \in U \times V \times W_3 : xyz = e\}$$

so

$$N_{UVW} = N_{UVW_1} + N_{UVW_2} + N_{UVW_3}.$$

Applying these observations, the statement follows as

$$\begin{aligned} N_{ABC} &= N_{ABG} - N_{ABA} - N_{ABB} = |A|.|B| - N_{BAA} - N_{BAB} \\ &= N_{BAG} - N_{BAA} - N_{BAB} = N_{BAC} = N_{CBA}. \end{aligned}$$

Problem 5. Let us define a subset $\mathcal{I}$ of the polynomial ring $\mathbb{R}[X]$ as follows:

$$\mathcal{I} = \Big\{P(X) = \sum_{j=0}^{m} b_j X^j : \sum_{j=0}^{m} b_j f(k + jl) = 0 \text{ for all } k, l \in \mathbb{Z}, l \neq 0\Big\}.$$

This is a subspace of the real vector space $\mathbb{R}[X]$. Furthermore, $P(X) \in \mathcal{I}$ implies $X.P(X) \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal, and it is non-zero, because the polynomial $R(X) = \sum\limits_{i=1}^{n} X^{a_i}$ belongs to $\mathcal{I}$. Thus, $\mathcal{I}$ is generated (as an ideal) by some non-zero polynomial Q.

If Q is constant then the definition of $\mathcal{I}$ implies $f = 0$, so we can assume that Q has a complex zero c. Again, by the definition of $\mathcal{I}$, the polynomial $Q(X^m)$ belongs to $\mathcal{I}$ for every natural number $m \geqslant 1$; hence $Q(X)$ divides $Q(X^m)$. This shows that all the complex numbers

$$c, c^2, c^3, c^4, \dots$$

are roots of Q. Since Q can have only finitely many roots, we must have $c^N = 1$ for some $N \geqslant 1$; in particular, $Q(1) = 0$, which implies $P(1) = 0$

for all $P \in \mathcal{I}$. This contradicts the fact that $R(X) = \sum\limits_{i=1}^{n} X^{a_i} \in \mathcal{I}$, and we are done.

Problem 6. We show that the number of nonzero coefficients can be $0, 1$ and 2. These values are possible, for example the polynomials $P_0(z) = 0, P_1(z) = 1$ and $P_2(z) = 1 + z$ satisfy the conditions and they have $0, 1$ and 2 nonzero terms, respectively.

Now consider an arbitrary polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfying the conditions and assume that it has at least two nonzero coefficients. Dividing the polynomial by a power of z and optionally replacing $p(z)$ by $-p(z)$, we can achieve $a_0 > 0$ such that conditions are not changed and the numbers of nonzero terms is preserved. So, without loss of generality, we can assume that $a_0 > 0$.

Let $Q(z) = a_1 z + \cdots + a_{n-1} z^{n-1}$. Our goal is to show that $Q(z) = 0$.

Consider those complex numbers $w_0, w_1, \dots, w_{n-1}$ on the unit circle for which $a_n w_k^n = |a_n|$; namely, let

$$w_k = \begin{cases} e^{\frac{2k\pi i}{n}} & \text{if } a_n > 0 \\ e^{\frac{(2k+1)\pi i}{n}} & \text{if } a_n < 0 \end{cases} \quad (k = 0, 1, \dots, n).$$

Notice that

$$\sum_{k=0}^{n-1} Q(w_k) = \sum_{k=0}^{n-1} Q\big(w_0 e^{\frac{2k\pi i}{n}}\big) = \sum_{j=1}^{n-1} a_j w_0^j \sum_{k=0}^{n-1} \Big(e^{\frac{2j\pi i}{n}}\Big)^k = 0.$$

Taking the average of polynomial $P(z)$ at the points w_k, we obtained

$$\frac{1}{n} \sum_{k=0}^{n-1} P(w_k) = \frac{1}{n} \sum_{k=0}^{n-1} (a_0 + Q(w_k) + a_n w_k^n) = a_0 + |a_n|$$

and

$$2 \geqslant \frac{1}{n} \sum_{k=0}^{n-1} |P(w_k)| \geqslant \left| \frac{1}{n} \sum_{k=0}^{n-1} P(w_k) \right| = a_0 + |a_n| \geqslant 2.$$

This obviously implies $a_0 = |a_n| = 1$ and $|P(w_k)| = |2 + Q(w_k)| = 2$ for all k. Therefore, all values of $Q(w_k)$ must lie on the circle $|2 + z| = 2$, while their sum is 0. This is possible only if $Q(w_k) = 0$ for all k. Then polynomial $Q(z)$ has at least n distinct roots while its degree is at most $n-1$. So $Q(z) = 0$ and $P(z) = a_0 + a_n z^n$ has only two nonzero coefficients.

Remark. From Parseval's formula (i.e. integrating $|P(z)|^2 = P(z)\overline{P(z)}$ on the unit circle) it can be obtained that

$$|a_0|^2 + \cdots + |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^2 dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} 4dt = 4. \tag{2.2}$$

Hence, there cannot be more than four nonzero coefficients, and if there are more than one nonzero term, then their coefficients are ± 1.

It is also easy to see that equality in (2.2) cannot hold two or more nonzero coefficients, so it is sufficient to consider only polynomials of the form $1 \pm x^m \pm x^n$. However, we do not know any simpler argument for these cases than the proof above.

2.14.2 Day 2

Problem 1. No. The function $f(x) = e^x$ also has this property since $ce^x = e^{x+\log c}$.

Problem 2. We claim that $29|x, y, z$. Then, $x^4 + y^4 + z^4$ is clearly divisible by 29^4.

Assume, to the contrary, that 29 does not divide all of the numbers x, y, z. Without loss of generality, we can suppose that $29 \nmid x$. Since the residue classes modulo 29 form a field, there is some $w \in \mathbb{Z}$ such that $xw \equiv 1 \pmod{29}$. Then $(xw)^4 + (yw)^4 + (zw)^4$ is also divisible by 29. So we can assume that $x \equiv 1 \pmod{29}$.

Thus , we need to show that $y^4 + z^4 \equiv -1 \pmod{29}$, i.e. $y^4 \equiv -1 - z^4 \pmod{29}$, is impossible. There are only eight fourth powers modulo 29,

$$\begin{aligned}
0 &\equiv 0^4, \\
1 &\equiv 1^4 \equiv 12^4 \equiv 17^4 \equiv 28^4 \pmod{29}, \\
7 &\equiv 8^4 \equiv 9^4 \equiv 20^4 \equiv 21^4 \pmod{29}, \\
16 &\equiv 2^4 \equiv 5^4 \equiv 24^4 \equiv 27^4 \pmod{29}, \\
20 &\equiv 6^4 \equiv 14^4 \equiv 15^4 \equiv 23^4 \pmod{29}, \\
23 &\equiv 3^4 \equiv 7^4 \equiv 22^4 \equiv 26^4 \pmod{29}, \\
24 &\equiv 4^4 \equiv 10^4 \equiv 19^4 \equiv 25^4 \pmod{29}, \\
25 &\equiv 11^4 \equiv 13^4 \equiv 16^4 \equiv 18^4 \pmod{29}.
\end{aligned}$$

The differences $-1-z^4$ are congruent to $28, 27, 21, 12, 8, 5, 4$ and 3. None of these residue classes is listed among the fourth powers.

Problem 3. Suppose $f(x) \neq x$ for all $x \in C$. Let $[a,b]$ be the smallest closed interval that contains C. Since C is closed, $a, b \in C$. By our hypothesis $f(a) > a$ and $f(b) < b$. Let $p = \sup\{x \in C : f(x) > x\}$. Since C is closed and f is continuous, $f(p) \geqslant p$, so $f(p) > p$. For all $x > p/, x \in C$ we have $f(x) < x$. Therefore $f\big(f(p)\big) < f(p)$ contrary to the fact that f is non-decreasing.

Problem 4. Notice that $A = B^2$, with $b_{ij} = \begin{cases} 1 & \text{if } i-j \equiv \pm 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$.
So it is sufficient to find $\det B$.

To find $\det B$, expand the determinant with respect to the first row, and then expad both terms with respect to the first columns.

$$\det B = \begin{vmatrix}
0 & 1 & & & & & 1 \\
1 & 0 & 1 & & & & \\
 & 1 & 0 & 1 & & & \\
 & & 1 & \ddots & \ddots & & \\
 & & & \ddots & 0 & 1 & \\
 & & & & 1 & 0 & 1 \\
1 & & & & & 1 & 0
\end{vmatrix}$$

$$
= -\begin{vmatrix}
1 & 1 & & & & \\
 & 0 & 1 & & & \\
 & 1 & \ddots & \ddots & & \\
 & & \ddots & 0 & 1 & \\
 & & & 1 & 0 & 1 \\
1 & & & & 1 & 0
\end{vmatrix}
+
\begin{vmatrix}
1 & 0 & 1 & & & \\
 & 1 & 0 & 1 & & \\
 & & 1 & \ddots & \ddots & \\
 & & & \ddots & 0 & 1 \\
 & & & & 1 & 0 \\
1 & & & & & 1
\end{vmatrix}
$$

$$
= -\left(
\begin{vmatrix}
0 & 1 & & & \\
1 & \ddots & \ddots & & \\
 & \ddots & 0 & 1 & \\
 & & 1 & 0 & 1 \\
 & & & 1 & 0
\end{vmatrix}
-
\begin{vmatrix}
1 & & & & \\
0 & 1 & & & \\
1 & \ddots & \ddots & & \\
 & \ddots & 0 & 1 & \\
 & & 1 & 0 & 1
\end{vmatrix}
\right) +
$$

$$
+\left(
\begin{vmatrix}
1 & 0 & 1 & & \\
 & 1 & \ddots & \ddots & \\
 & & \ddots & 0 & 1 \\
 & & & 1 & 0 \\
 & & & & 1
\end{vmatrix}
-
\begin{vmatrix}
0 & 1 & & & \\
1 & 0 & 1 & & \\
 & 1 & \ddots & \ddots & \\
 & & \ddots & 0 & 1 \\
 & & & 1 & 0
\end{vmatrix}
\right) = -(0-1)+(1-0) = 2,
$$

since the second and the third matrices are lower/upper triangular, while in the first and the fourth matrices we have $\mathrm{row}_1 - \mathrm{row}_3 + \mathrm{row}_5 - \cdots \pm \mathrm{row}_{n-2} = \overline{0}$.

So $\det B = 2$ and thus $\det A = 4$.

Problem 5. The answer is $n_k = 2^k$. In that case, the matrices can be constructed as follows: Let V be the n-dimensional real vector space with basis elements $[S]$, where S runs through all $n = 2^k$ subsets of $\{1, 2, \ldots, k\}$. Define A_i as an endomorphism of V by

$$A_i[S] = \begin{cases} 0 & \text{if } i \in S \\ [S \cup \{i\}] & \text{if } i \notin S \end{cases}$$

for all $i = 1, 2, \ldots, k$ and $S \subset \{1, 2, \ldots, k\}$. Then $A_i^2 = 0$ and $A_iA_j = A_jA_i$. Furthermore,

$$A_1A_2 \ldots A_k[\emptyset] = [\{1, 2, \ldots, k\}],$$

and hence $A_1A_2 \ldots A_k \neq 0$.

Now let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices satisfying the conditions of the problem; we prove that $n \geqslant 2^k$. Let v be a real vector satisfying

$A_1A_2\dots A_kv \neq 0$. Denote by $\mathcal{P}$ the set of all subsets of $\{1,2,\dots,k\}$. Choose a complete ordering $\prec$ on $\mathcal{P}$ with the property

$$X \prec Y \Rightarrow |X| \leqslant |Y| \text{ for all } X, Y \in \mathcal{P}.$$

For every element $X = \{x_1, x_2, \dots, x_r\} \in \mathcal{P}$, define $A_X = A_{x_1}A_{x_2}\dots A_{x_r}$ and $v_X = A_Xv$. Finally, write $\overline{X} = \{1,2,\dots,k\} \setminus X$ for the complement of X.

Now take $X, Y \in \mathcal{P}$ with $X \precneqq Y$. Then $A_{\overline{X}}$ annihilates v_Y, because $X \precneqq Y$ implies the existence of some $y \in Y \setminus X = Y \cap \overline{X}$, and

$$A_{\overline{X}}v_Y = A_{\overline{X}\setminus\{y\}}A_yA_yv_{Y\setminus\{y\}} = 0,$$

since $A_y^2 = 0$. So $A_{\overline{X}}$ annihilates the span of all the v_Y with $X \precneqq Y$. This implies that v_X does not lie in this span, because $A_{\overline{X}}v_X = v_{\{1,2,\dots,k\}} \neq 0$. Therefore, the vectors v_X (with $X \in \mathcal{P}$) are linearly independent; hence $n \geqslant |\mathcal{P}| = 2^k$.

Problem 6. For the proof, we need the following

Lemma 1. For any polynomial g, denote by $d(g)$ the minimum distance of any two of its real zeros ($d(g) = \infty$ if g has at most one real zero). Assume that g and $g + g'$ both are of degree $k \geqslant 2$ and have k distinct real zeros. Then $d(g + g') \geqslant d(g)$.

Proof of Lemma 1: Let $x_1 < x_2 < \dots < x_k$ be the roots of g. Suppose a, b are roots of $g + g'$ satisfying $0 < b - a < d(g)$. Then a, b cannot be roots of g and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \tag{2.3}$$

Since $\dfrac{g'}{g}$ is strictly decreasing between consecutive zeros of g, we must have $a < x_j < b$ for some j.

For all $i = 1, 2, \dots, k-1$ we have $x_{i+1} - x_i > b - a$, hence $a - x_i > b - x_{i+1}$. If $i < j$, both sides of this inequality are negative; if $i \geqslant j$, both sides are positive. In any case, $\dfrac{1}{a-x_i} < \dfrac{1}{b-x_{i+1}}$, and hence

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a-x_i} + \underbrace{\frac{1}{a-x_k}}_{<0} < \sum_{i=1}^{k-1} \frac{1}{b-x_{i+1}} + \underbrace{\frac{1}{b-x_1}}_{>0} = \frac{g'(b)}{g(b)}$$

This contrdicts (2.3).

Now we turn to the proof of the stated problem. Denote by m the degree of f. We will prove by induction on m that f_n has m distinct real zeros for sufficient large n. The cases $m = 0, 1$ are trivial; so we assume $m \geqslant 2$. Without loss of generality we can assume that f is monic. By induction, the result holds for f', and by ignoring the first few terms we can assume that f_n' has $m-1$ distinct real zeros for all n. Let denote these zeros by $x_1^{(n)} > x_2^{(n)} > \cdots > x_{m-1}^{(n)}$. Then f_n has minima in $x_1^{(n)}, x_3^{(n)}, x_5^{(n)}, \ldots$, and maxima in $x_2^{(n)}, x_4^{(n)}, x_6^{(n)}, \ldots$. Note that in the interval $(x_{i+1}^{(n)}, x_i^{(n)})$, the function $f_{n+1}' = f_n' + f_n''$ must have a zero (this follows by applying Rolle's theorem to the function $e^x f_n'(x)$); the same is true for the interval $(-\infty, x_{m-1}^{(n)})$. Hence, in each of these $m-1$ intervals, f_{n+1}' has exactly one zero. This shows that

$$x_1^{(n)} > x_1^{(n+1)} > x_2^{(n)} > x_2^{(n+1)} > x_3^{(n)} > x_3^{(n+1)} > \ldots \tag{2.4}$$

Lemma 2. We have $\lim\limits_{n\to\infty} f_n\big(x_j^{(n)}\big) = -\infty$ if j is odd, and $\lim\limits_{n\to\infty} f_n\big(x_j^{(n)}\big) = +\infty$ if j is even.

Lemma 2 immediately implies the result: For sufficiently large n, the values of all maxima of f_n are positive, and the values of all minima of f_n are negative; this implies that f_n has m distinct zeros.

Proof of Lemma 2: Let $d = \min\{d(f'), 1\}$; then by Lemma 1, $d(f_n') \geqslant g$ for all n. Define $\epsilon = \dfrac{(m-1)d^{m-1}}{m^{m-1}}$; we will show that

$$f_{n+1}\big(x_j^{(n+1)}\big) \geqslant f_n\big(x_j^{(n)}\big) + \epsilon \ \text{ for } j \text{ even.} \tag{2.5}$$

(The corresponding result for odd j can be shown similarly.) Do to so, write $f = f_n, b = x_j^{(n)}$, and choose a satisfying $d \leqslant b - a \leqslant 1$ such that

f' has no zero inside (a,b). Define ξ by the relation $b-\xi=\dfrac{1}{m}(b-a)$; then $\xi\in(a,b)$. We show that $f(\xi)+f'(\xi)\geqslant f(b)+\epsilon$.

Notice that

$$\begin{aligned}
\frac{f''(\xi)}{f'(\xi)} &= \sum_{i=1}^{m-1}\frac{1}{\xi-x_i^{(n)}}\\
&= \sum_{i<j}\underbrace{\frac{1}{\xi-x_i^{(n)}}}_{<\frac{1}{\xi-a}}+\frac{1}{\xi-b}+\sum_{i>j}\underbrace{\frac{1}{\xi-x_i^{(n)}}}_{<0}\\
&< (m-1)\frac{1}{\xi-a}+\frac{1}{\xi-b}=0.
\end{aligned}$$

The last equality holds by definition of ξ. Since f' is positive and $\dfrac{f''}{f'}$ is decreasing in (a,b), we have that f'' is negative on (ξ,b). Therefore,

$$f(b)-f(\xi)=\int_\xi^b f'(t)dt\leqslant\int_\xi^b f'(\xi)dt=(b-\xi)f'(\xi).$$

Hence,

$$\begin{aligned}
f(\xi)+f'(xi) &\geqslant f(b)-(b-\xi)f'(\xi)+f'(\xi)\\
&= f(b)+(1-(\xi-b))f'(\xi)\\
&= f(b)+\big(1-\frac{1}{m}(b-a)\big)f'(\xi)\\
&\geqslant f(b)+\big(1-\frac{1}{m}\big)f'(\xi).
\end{aligned}$$

Together with

$$f'(\xi)=|f'(\xi)|=m\prod_{i=1}^{m-1}\underbrace{|\xi-x_i^{(n)}|}_{\geqslant|\xi-b|}\geqslant m|\xi-b|^{m-1}\geqslant\frac{d^{m-1}}{m^{m-1}}$$

we get

$$f(\xi)+f'(\xi)\geqslant f(b)+\epsilon.$$

Together with (2.4) this shows (2.5). This finishes the proof of Lemma 2.

2.15 Solutions of Olympic 2008

2.15.1 Day 1

Problem 1. We prove that $f(x) = ax + b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{R}$. These functions obviously satify the conditions.

Suppose that a function $f(x)$ fulfills the required properties. For an arbitrary rational q, consider the function $g_q(x) = f(x+q) - f(x)$. This is a continuous function which attains only rational values, therefore g_q is constant.

Set $a = f(1) - f(0)$ and $b = f(0)$. Let n be an arbitrary positive integer and let $r = f\big(\frac{1}{n}\big) - f(0)$. Since $f\big(x + \frac{1}{n}\big) - f(x) = f\big(\frac{1}{n}\big) - f(0) = r$ for all x, we have

$$f\Big(\frac{k}{n}\Big) - f(0) = \Big(f\big(\frac{1}{n}\big) - f(0)\Big) + \Big(f\big(\frac{2}{n}\big) - f\big(\frac{1}{n}\big)\Big) + \dots + \Big(f\big(\frac{k}{n}\big) - f\big(\frac{k-1}{n}\big)\Big) = kr$$

and

$$f\big(-\frac{k}{n}\big) - f(0) = -\Big(f(0) - f\big(-\frac{1}{n}\big)\Big) - \Big(f\big(-\frac{1}{n}\big) - f\big(-\frac{2}{n}\big)\Big) - \dots - \Big(f\big(-\frac{k-1}{n}\big) - f\big(-\frac{k}{n}\big)\Big) = -kr \quad \text{for} \quad k \geqslant 1.$$

In the case $k = n$ we get $a = f(1) - f(0) = nr$, so $r = \frac{a}{n}$. Hence, $f\big(\frac{k}{n}\big) - f(0) = kr = \frac{ak}{n}$ and then $f\big(\frac{k}{n}\big) = a.\frac{k}{n} + b$ for all integer k and $n > 0$.

So, we have $f(x) = ax + b$ for all rational x. Since the function f is continuous and the rational numbers form a dense subset of $\mathbb{R}$, the same holds for all real x.

Problem 2. We can assume that $P \neq 0$.

Let $f \in V$ be such that $P(f) \neq 0$. Then $P(f^2) \neq 0$, and therefore $P(f^2) = aP(f)$ for some non-zero real a. Then $0 = P(f^2 - af) = P(f(f-a))$ implies $P(f-a) = 0$, so we get $P(a) \neq 0$. By rescaling, we can assume that $P(1) = 1$. Now $P(X+b) = 0$ for $b = -P(X)$. Replacing P by $\widehat{P}$ given as

$$\widehat{P}(f(X)) = P(f(X+b))$$

we can assume that $P(X) = 0$.

Now we are going to prove that $P(X^k) = 0$ for all $k \geqslant 1$. Suppose this is true for all $k < n$. We know that $P(X^n + e) = 0$ for $e = -P(X^n)$. From the induction hypothesis we get

$$P((X+e)(X+1)^{n-1}) = P(X^n + e) = 0,$$

and therefore $P(X+e) = 0$ (since $P(X+1) = 1 \neq 0$). Hence $e = 0$ and $P(X^n) = 0$, which completes the inductive step. From $P(1) = 1$ and $P(X^k) = 0$ for $k \geqslant 1$ we immediately get $P(f) = f(0)$ for all $f \in V$.

Problem 3. The theorem is obvious if $p(a_i) = 0$ for some i, so assume that all $p(a_i)$ are nonzero and pairwise different.

There exist numbers s, t such that $s|p(a_1), t|p(a_2), st = lcm(p(a_1), p(a_2))$ and $gsd(s,t) = 1$.

As s, t are relatively prime numbers, there exist $m, n \in \mathbb{Z}$ such that $a_1 + sn = a_2 + tm =: b_2$. Obviously $s|p(a_1 + sn) - p(a_1)$ and $t|p(a_2 + tm) - p(a_2)$, so $st|p(b_2)$.

Similarly one obtains b_3 such that $p(a_3)|p(b_3)$ and $p(b_2)|p(b_3)$ thus also $p(a_1)|p(b_3)$ and $p(a_2)|p(b_3)$.

Reasoning inductively we obtain the existence of $a = b_k$ as required.

The polynomial $p(x) = 2x^2 + 2$ shows that the second part of the problem is not true, as $p(0) = 2, p(1) = 4$ but no value of $p(a)$ is divisible by 8 for integer a.

Remark. One can assume that the $p(a_i)$ are nonzero and ask for a such that $p(a)$ is a nonzero multiple of all $p(a_i)$. In the solution above, it can happen that $p(a) = 0$. But every number $p(a + np(a_1)p(a_2)\dots p(a_k))$ is also divisible by every $p(a_i)$, since the polynomial is nonzero, there exists n such that $p(a + np(a_1)p(a_2)\dots p(a_k))$ satisfies the modified thesis.

Problem 4. The answer is $n \geqslant 4$.

Consider the following set of special triples:

$$\left(0, \frac{8}{15}, \frac{7}{15}\right), \ \left(\frac{2}{5}, 0, \frac{3}{5}\right), \ \left(\frac{3}{5}, \frac{2}{5}, 0\right), \ \left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right).$$

We will prove that any special triple (x, y, z) is worse than one of these (triple a is worse than triple b if triple b is better than triple a). We suppose that some special triple (x, y, z) is actually not worse than the first three of the triples from the given set, derive some conditions on x, y, z and prove that, under these conditions, (x, y, z) is worse than the fourth triple from the set.

Triple (x, y, z) is not worse than $\left(0, \frac{8}{15}, \frac{7}{15}\right)$ means that $y \geqslant \frac{8}{15}$ or $z \geqslant \frac{7}{15}$. Triple (x, y, z) is not worse than $\left(\frac{2}{5}, 0, \frac{3}{5}\right)$ — $x \geqslant \frac{2}{5}$ or $z \geqslant \frac{3}{5}$. Triple (x, y, z) is not worse than $\left(\frac{3}{5}, \frac{2}{5}, 0\right)$ — $x \geqslant \frac{3}{5}$ or $z \geqslant \frac{2}{5}$. Since $x + y + z = 1$, then it is impossible that all inequalities $x \geqslant \frac{2}{5}, y \geqslant \frac{2}{5}$ and $z \geqslant \frac{7}{15}$ are true. Suppose that $x < \frac{2}{5}$, then $y \geqslant \frac{2}{5}$ and $z \geqslant \frac{3}{5}$. Using $x + y + z = 1$ and $x \geqslant 0$ we get $x = 0, y = \frac{2}{5}, z = \frac{3}{5}$. We obtain the triple $\left(0, \frac{2}{5}, \frac{3}{5}\right)$ which is worse than $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$. Suppose that $y < \frac{2}{5}$, then $x \geqslant \frac{3}{5}$ and $y \geqslant \frac{7}{15}$ and this is a contradiction to the admissibility of (x, y, z). Suppose that $z < \frac{7}{15}$, then $x \geqslant \frac{2}{5}$ and $y \geqslant \frac{8}{15}$. We get (by admissibility, again) that $z \leqslant \frac{1}{15}$ and $y \leqslant \frac{3}{5}$. The last inequalities imply that $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$ is better than (x, y, z).

We will prove that for any given set of three special triples one can find a special triple which is not worse than any triple from the set. Suppose we have a set S of three special triples

$$(x_1, y_1, z_1), \ (x_2, y_2, z_2), \ (x_3, y_3, z_3).$$

Denote

$$a(S) = \min(x_1, x_2, x_3), b(S) = \min(y_1, y_2, y_3), c(S) = \min(z_1, z_2, z_3).$$

It is easy to check that S_1:

$$\left(\frac{x_1 - a}{1-a-b-c}, \frac{y_1 - b}{1-a-b-c}, \frac{z_1 - c}{1-a-b-c}\right)$$
$$\left(\frac{x_2 - a}{1-a-b-c}, \frac{y_2 - b}{1-a-b-c}, \frac{z_2 - c}{1-a-b-c}\right)$$
$$\left(\frac{x_3 - a}{1-a-b-c}, \frac{y_3 - b}{1-a-b-c}, \frac{z_3 - c}{1-a-b-c}\right)$$

is a set of three special triples also (we may suppose that $a+b+c<1$, because otherwise all three triples are equal and our statement is trivial).

If there is a special triple (x, y, z) which is not worse than any triple from S_1, then the triple

$$((1-a-b-c)x + a, (1-a-b-c)y + b, (1-a-b-c)z + c)$$

is special and not worse than any triple from S. We also have $a(S_1) = b(S_1) = c(S_1) = 0$, so we may suppose that the same holds for our starting set S.

Suppose that one element of S has two entries equal to 0.

Note that one of the two remaining triples from S is not worse than the other. This triple is also not worse than all triples from S because any special triple is not worse than itself and the triple with two zeroes.

So we have $a = b = c = 0$ but we may suppose that all triples from S contain at most one zero. By transposing triples and elements in triples (elements in all triples must be transposed simultaneously) we may achieve the following situation $x_1 = y_2 = z_3 = 0$ and $x_2 \geqslant x_3$. If $z_2 \geqslant z_1$, then the second triple $(x_2, 0, z_2)$ is not worse than the other two triples from S. So we may assume that $z_1 \geqslant z_2$. If $y_1 \geqslant y_3$, then the first triple is not worse than the second and the third and we assume $y_3 \geqslant y_1$. Consider the three pairs of numbers $x_2, y_1; z_1, x_3; y_3, z_2$. The sum of all these numbers is three and consequently the sum of the numbers in one of the pairs is less than or equal to one. If it is the first pair then the triple $(x_2, 1 - x_2, 0)$ is not worse than all triples from S, for the second we may take $(1-z_1, 0, z_1)$ and for the third — $(0, y_3, 1-y_3)$. So we found a desirable special triple for any given S.

Problem 5. Yes. Let H be the commutative group $H = \mathbb{F}_2^3$, where $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the field with two elements. The group of automorphisms of H is the general linear group $GL_3\mathbb{F}_2$; it has

$$(8-1).(8-2).(8-4) = 7.6.4 = 168$$

elements. One of them is the shift operator $\phi : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$.

Now let $T = \{a^0, a^1, a^2\}$ be a group of order 3 (written multiplicatively); it acts on H by $\tau(a) = \phi$. Let G be the semidirect product $G = H \rtimes_\tau T$. In other words, G is the group of 24 elements

$$G = \{ba^i : b \in H, i \in (\mathbb{Z}/3\mathbb{Z})\}, \;\; ab = \phi(b)a.$$

G has one element e of order 1 and seven elements $b, b \in H, b \neq e$ of order 2.

If $g = ba$, we find that $g^2 = baba = b\phi(b)a^2 \neq e$, and that

$$g^3 = b\phi(b)a^2ba = b\phi(b)a\phi(b)a^2 = b\phi(b)\phi^2(b)a^3 = \psi(b),$$

where the homomorphism $\psi : H \to H$ is defined as $\psi : (x_1, x_2, x_3) \mapsto (x_1+x_2+x_3)(1, 1, 1)$. It is clear that $g^3 = \psi(b) = e$ for 4 elements $b \in H$, while $g^6 = \psi^2(b) = e$ for all $b \in H$.

We see that G has 8 elements of order 3, namely ba and ba^2 with $b \in \ker\psi$, and 8 elements of order 6 namely ba and ba^2 with $b \notin \ker\psi$. That accounts for orders of all elements of G.

Let $b_0 \in H \setminus \ker\psi$ be arbitrary; it is easy to see that G is generated by b_0 and a. As every automorphism of G is fully determined by its action on b_0 and a, it follows that G has no more than $7.8 = 56$ automorphisms.

Remark. G and H can be equivalently presented as subgroups of S_6, namely as $H = \big\langle (135)(246), (12) \big\rangle$.

Problem 6. Consider the $n \times n$ determinant

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-1} \\ x & 1 & \dots & x^{n-2} \\ \vdots & \vdots & \dots & \vdots \\ x^{n-1} & x^{n-2} & \dots & 1 \end{vmatrix}$$

where the ij-th entry is $x^{|i-j|}$. From the definition of the determinant we get

$$\Delta(x) = \sum_{(i_1,i_2,\dots,i_n)\in S_n} (-1)^{inv(i_1,i_2,\dots,i_n)} x^{D(i_1,i_2,\dots,i_n)}$$

where S_n is the set of all permutations of $(1,2,\dots,n)$ and $inv(i_1,i_2,\dots,i_n)$ denotes the number of inversions in the sequence $(i_1,i_2,\dots,i_n)$. So $Q(n,d)$ has the same parity as the coefficients of x^d in $\Delta(x)$.

It remains to evaluate $\Delta(x)$. In order to eliminate the entries below the diagonal, subtract the $(n-1)$-th row, multipled by x, from the n-th row. Then subtract the $(n-2)$-th row, multipled by x, from the $(n-1)$-th and so on. Finally, subtract the first row, multipled by x, from the second row.

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ x & 1 & \dots & x^{n-3} & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & \dots & 1 & x \\ x^{n-1} & x^{n-2} & \dots & x & 1 \end{vmatrix}$$

$$= \dots = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ 0 & 1-x^2 & \dots & x^{n-3}-x^{n-1} & x^{n-2}-x^n \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1-x^2 & x-x^2 \\ 0 & 0 & \dots & 0 & 1-x^2 \end{vmatrix} = (1-x^2)^{n-1}.$$

For $d \geqslant 2n$, the coefficient of x^d is 0 so $Q(n,d)$ is even.

2.15.2 Day 2

Problem 1. Let $f(x) = x^{2n} + x^n + 1, g(x) = x^{2k} - x^k + 1, h(x) = x^{2k} + x^k + 1$. The complex number $x_1 = \cos\left(\frac{\pi}{3k}\right) + i\sin\left(\frac{\pi}{3k}\right)$ is a root of $g(x)$.

Let $\alpha = \frac{\pi n}{3k}$. Since $g(x)$ divides $f(x), f(x_1) = g(x_1) = 0$. So, $0 = x_1^{2n} + x_1^n + 1 = \big(\cos(2\alpha) + i\sin(2\alpha)\big) + \big(\cos\alpha + i\sin\alpha\big) + 1 = 0$, and $(2\cos\alpha + 1)\big(\cos\alpha + i\sin\alpha\big) = 0$. Hence $2\cos\alpha + 1 = 0$, i.e. $\alpha = \pm\frac{2\pi}{3} + 2\pi c$, where $c \in \mathbb{Z}$.

Let x_2 be a root of the polynomial $h(x)$. Since $h(x) = \frac{x^{3k}-1}{x^k-1}$, the roots of the polynomial $h(x)$ are distinct and they are $x_2 = \cos\frac{2\pi s}{3k} + i\sin\frac{2\pi s}{3k}$, where $s = 3a \pm 1, a \in \mathbb{Z}$. It is enough to prove that $f(x_2) = 0$. We have $f(x_2) = x_2^{2n}+x_2^n+1 = \big(\cos(4s\alpha)+\sin(4s\alpha)\big)+\big(\cos(2s\alpha)+\sin(2s\alpha)\big)+1 = \big(2\cos(2s\alpha) + 1\big)\big(\cos(2s\alpha) + i\sin(2s\alpha)\big) = 0$ (since $2\cos(2s\alpha) + 1 = 2\cos\big(2s(\pm\frac{2\pi}{3} + 2\pi c)\big) + 1 = 2\cos\big(\frac{4\pi s}{3}\big) + 1 = 2\cos\big(\frac{4\pi}{3}(3a \pm 1)\big) + 1 = 0$).

Problem 2. It is well known that an ellipse might be defined by a focus (a point) and a directrix (a straight line), as a locus of points such that the distance to the focus divided by the distance to directrix is equal to a given number $e < 1$. So, if a point X belongs to both ellipses with the same focus F and directrices $l_1.l_2$, then $e_1.l_1X = FX = e_2.l_2X$ (here we denote by l_1X, l_2X distances between the corresponding line and the point X). The equation $e_1.l_1X = e_2.l_2X$ defines two lines whose equations are linear combinations with coefficients $e_1, \pm e_2$ of the normalized equations of lines l_1, l_2 but of those two only one is relevant, since X and F should lie on the same side of each directrix. So, we have that all possible points lie on one line. The intersection of a line and an ellipse consists of at most two points.

Problem 3. As is known, the Fibonacci numbers F_n can be expressed as $F_n = \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^n - \big(\frac{1-\sqrt{5}}{2}\big)^n\Big)$. Expanding this expression, we obtain that $F_n = \frac{1}{2^{n-1}}\Big(\binom{n}{1} + \binom{n}{3}5 + \cdots + \binom{n}{l}5^{\frac{l-1}{2}}\Big)$, where l is the greatest odd numbers such that $l \leqslant n$ and $s = \frac{l-1}{2} \leqslant \frac{n}{2}$.

So, $F_n = \frac{1}{2^{n-1}}\sum\limits_{k=0}^{s}\binom{n}{2k+1}5^k$, which implies that 2^{n-1} divides $\sum\limits_{0\leqslant k\leqslant\frac{n}{2}}\binom{n}{2k+1}5^k$.

Problem 4. Let $f(x) = g(x)h(x)$ where $h(x)$ is a polynomial with integer coefficients.

Let $a_1, \ldots, a_{81}$ be distinct integer roots of the polynomial $f(x) -$

2008. Then $f(a_i) = g(a_i)h(a_i) = 2008$ for $i = 1, \ldots, 81$. Hence, $g(a_1), \ldots, g(a_{81})$ are integer divisors of 2008.

Since $2008 = 2^3.251$ $(2, 251$ are primes) then 2008 has exactly 16 distinct integer divisors (including the negative divisors as well). By the pigeonhole principle, there are at least 6 equal numbers among $g(a_1), \ldots, g(a_{81})$ (because $81 > 16.5$). For example, $g(a_1) = g(a_2) = \cdots = g(a_6) = c$. So $g(x) - c$ is a nonconstant polymial which has at least 6 distinct roots (namely $a_1, \ldots, a_6$). Then the degree of the polynomial $g(x) - c$ is at least 6.

Problem 5. Call a square matrix of type (B), if it is of the form

$$\begin{pmatrix} 0 & b_{12} & 0 & \ldots & b_{1,2k-2} & 0 \\ b_{21} & 0 & b_{23} & \ldots & 0 & b_{2,2k-1} \\ 0 & b_{32} & 0 & \ldots & b_{3,2k-2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2k-2,1} & 0 & b_{2k-2,3} & \ldots & 0 & b_{2k-2,2k-1} \\ 0 & b_{2k-1,2} & 0 & \ldots & b_{2k-1,2k-2} & 0 \end{pmatrix}$$

Note that every matrix of this form has determinant zero, because it has k columns spanning a vector space of dimension at most $k-1$.

Call a square matrix of type (C), if it is of the form

$$C' = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & \ldots & 0 & c_{1,k} \\ c_{11} & 0 & c_{12} & 0 & \ldots & c_{1,k} & 0 \\ 0 & c_{21} & 0 & c_{22} & \ldots & 0 & c_{2,k} \\ c_{21} & 0 & c_{22} & 0 & \ldots & c_{2,k} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{k,1} & 0 & c_{k,2} & \ldots & 0 & c_{k,k} \\ c_{k,1} & 0 & c_{k,2} & 0 & \ldots & c_{k,k} & 0 \end{pmatrix}$$

By permutations of rows and columns, we see that

$$|\det C'| = \left| \det \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right| = |\det C|^2,$$

where C denotes the $k \times k$-matrix with coefficients $c_{i,j}$. Therefore, the determinant of any matrix of type (C) is a perfect square (up to a sign).

Now let X' be the matrix obtained from A by replacing the first row by $(1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0)$, and let Y be the matrix obtained from A by replacing the entry a_{11} by 0. By multi-linearity of the determinant, $\det(A) = \det(X') + \det(Y)$. Note that X' can be written as

$$X' = \begin{pmatrix} 1 & 0 \\ v & X \end{pmatrix}$$

for some $(n-1) \times (n-1)$-matrix X and some column vector v. Then $\det(A) = \det(X) + \det(Y)$. Now consider two cases. If n is old, then X is of type (C), and Y is of type (B). Therefore, $|\det(A)| = |\det(X)|$ is a perfect square. If n is even, then X is of type (B), and Y is of type (C); hence $|\det(A)| = |\det(Y)|$ is a perfect square.

The set of primes can be replaced by any subset of $\{2\} \cup \{3, 5, 7, 9, 11, \dots\}$.

Problem 6. It is clear that, if $\mathcal{B}$ is an orthonormal system in a Hilbert space $\mathcal{H}$, then $\{\frac{d}{\sqrt{2}}e : e \in \mathcal{B}\}$ is a set of points in $\mathcal{H}$, any two of which are at distance d apart. We need to show that every set S of equidistant points is a translate of such a set.

We begin by noting that, if $x_1, x_2, x_3, x_4 \in S$ are four distinct points, then

$$\begin{aligned}
\langle x_2 - x_1, x_2 - x_1 \rangle &= d^2, \\
\langle x_2 - x_1, x_3 - x_1 \rangle &= \frac{1}{2}\left(\|x_2 - x_1\|^2 + \|x_3 - x_1\|^2 - \|x_2 - x_3\|^2\right) = \frac{1}{2}d^2, \\
\langle x_2 - x_1, x_4 - x_3 \rangle &= \langle x_2 - x_1, x_4 - x_1 \rangle - \langle x_2 - x_1, x_3 - x_1 \rangle \\
&= \frac{1}{2}d^2 - \frac{1}{2}d^2 = 0.
\end{aligned}$$

This shows that scalar products among vectors which are finite linear combinations of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n,$$

where $x_1, x_2, \dots, x_n$ are distinct points in S and $\lambda_1, \lambda_2, \dots \lambda_n$ are integers with $\lambda_1 + \lambda_1 + \cdots + \lambda_1 = 0$, are universal across all such sets S in all Hilbert spaces $\mathcal{H}$; in particular, we may conveniently evaluate them using examples of our choosing, such as the canonical example above in $\mathbb{R}^n$. In fact this property trivially follows also when coefficients λ_i are rational, and hence by continuity any real numbers with sum 0.

If $S = \{x_1, x_2, \dots, x_n\}$ is a finite set, we form

$$x = \frac{1}{n}(x_1 + x_2 + \dots + x_n),$$

pick a nonzero vector $z \in [\mathrm{Span}(x_1 - x, x_2 - x, \dots, x_n - x)]^\perp$ and seek y in the form $y = x + \lambda z$ for a suitable $\lambda \in \mathbb{R}$. We find that

$$\big\langle x_1 - y, x_2 - y\big\rangle = \big\langle x_1 - x - \lambda z, x_2 - x - \lambda z\big\rangle = \big\langle x_1 - x, x_2 - x\big\rangle + \lambda^2 \|z\|^2,$$

$\big\langle x_1 - x, x_2 - x\big\rangle$ may be computed by our remark above as

$$\begin{aligned}\big\langle x_1 - x, x_2 - x\big\rangle &= \frac{d^2}{2}\left\langle \Big(\frac{1}{n} - 1, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\Big)^\perp, \Big(\frac{1}{n}, \frac{1}{n} - 1, \frac{1}{n}, \dots, \frac{1}{n}\Big)^\perp \right\rangle_{\mathbb{R}^n} \\ &= \frac{d^2}{2}\Big(\frac{2}{n}\Big(\frac{1}{n} - 1\Big) + \frac{n-2}{n^2}\Big) = -\frac{d^2}{2n}.\end{aligned}$$

So the choice $\lambda = \dfrac{d}{\sqrt{2n}\|z\|}$ will make all vectors $\dfrac{\sqrt{2}}{d}(x_i - y)$ orthogonal to each other; it is easily checked as above that they will also be of length one.

Let now S be an infinite set. Pick an infinite sequence

$$T = \{x_1, x_2, \dots, x_n, \dots\}$$

of distinct points in S. We claim that the sequence

$$y_n = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$$

is a Cauchy sequence in $\mathcal{H}$. (This is the crucial observation). Indeed, for $m > n$, the norm $\|y_m - y_n\|$ may be computed by the above remark as

$$\begin{aligned}\|y_m - y_n\|^2 &= \frac{d^2}{2}\Big\|\Big(\frac{1}{m} - \frac{1}{n}, \dots, \frac{1}{m} - \frac{1}{n}, \frac{1}{m}, \frac{1}{m}\Big)^\top\Big\|^2_{\mathbb{R}^m} \\ &= \frac{d^2}{2}\Big(\frac{n(m-n)^2}{m^2 n^2} + \frac{m-n}{m^2}\Big) \\ &= \frac{d^2}{2}\frac{(m-n)(m-n+n)}{m^2 n} = \frac{d^2}{2}\frac{m-n}{mn} \\ &= \frac{d^2}{2}\Big(\frac{1}{n} - \frac{1}{m}\Big) \to 0, \quad m, n \to \infty.\end{aligned}$$

By completeness of $\mathcal{H}$, it follows that there exists a limit

$$y = \lim_{n\to\infty} y_n \in \mathcal{H}.$$

We claim that y satisfies all conditions of the problem. For $m > n > p$, with n, p fixed, we compute

$$\begin{aligned}\|x_n - y_m\|^2 &= \frac{d^2}{2}\Big\|\Big(-\frac{1}{m},\dots,-\frac{1}{m},1-\frac{1}{m},-\frac{1}{m},\dots,-\frac{1}{m}\Big)^\top\Big\|^2_{\mathbb{R}^m}\\ &= \frac{d^2}{2}\Big[\frac{m-1}{m^2}+\frac{(m-1)^2}{m^2}\Big] = \frac{d^2}{2}\frac{m-1}{m} \to \frac{d^2}{2},\ \ m\to\infty,\end{aligned}$$

showing that $\|x_n - y\| = \frac{d}{\sqrt{2}}$, as well as

$$\begin{aligned}\big\langle x_n - y_m, x_p - y_m\big\rangle &= \frac{d^2}{2}\Big\langle\Big(-\frac{1}{m},\dots,-\frac{1}{m},\dots,1-\frac{1}{m},\dots,-\frac{1}{m}\Big)^\perp,\\ &\qquad \Big(-\frac{1}{m},\dots,1-\frac{1}{m},\dots,-\frac{1}{m},\dots,-\frac{1}{m}\Big)^\perp\Big\rangle_{\mathbb{R}^m}\\ &= \frac{d^2}{2}\Big[\frac{m-2}{m^2}-\frac{2}{m}\big(1-\frac{1}{m}\big)\Big] = -\frac{d^2}{2m}\to 0,\ \ m\to\infty,\end{aligned}$$

showing that $\big\langle x_n - y, x_p - y\big\rangle = 0$, so that

$$\left\{\frac{\sqrt{2}}{d}(x_n - y) : n \in \mathbb{N}\right\}$$

is indeed an orthonormal system of vectors.

This completes the proof in case when $T = S$, which we can always take if S is countable. If it is not, let x', x'' be any two distinct points in $S \setminus T$. Then applying the above procedure to the set

$$T' = \{x', x'', x_1, x_2, \dots, x_n, \dots\}$$

it follows that

$$\lim_{n\to\infty}\frac{x' + x'' + x_1 + x_2 + \cdots + x_n}{n+2} = \lim_{n\to\infty}\frac{x_1 + x_2 + \cdots + x_n}{n} = y$$

satisfies that

$$\left\{\frac{\sqrt{2}}{d}(x' - y), \frac{\sqrt{2}}{d}(x'' - y)\right\} \cup \left\{\frac{\sqrt{2}}{d}(x_n - y) : n \in \mathbb{N}\right\}$$

is still an orthonormal system.

This it true for any distinct $x', x'' \in S \setminus T$; it follows that the entire system

$$\left\{ \frac{\sqrt{2}}{d}(x-y) : x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$, as required.